On the Existence of the *tt*-Mitotic Hypersimple Set Which is not *btt*-Mitotic

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ABSTRACT
Let us adduce some definitions:
If a recursively enumerable (r.e.) set A is a disjoint union of two sets B and C, then we say that B, C is an *r.e. splitting of* A.
The r.e. set A is *tt-mitotic* (*btt-mitotic*) if there is an r.e. splitting (B,C) of A such that the sets B and C both belong to the same tt- (btt-) degree of unsolvability, as the set A.
In this paper the existence of the tt-mitotic hypersimple set, which is not btt-mitotic is proved.

Keywords
Recursively enumerable (r.e.) set, hypersimple set, mitotic set, tt-reducibility, btt-reducibility.

1. INTRODUCTION
Notation. We shall use the notions and terminology introduced in (Soare [6]), (Downey and Stob [1]), (Rogers [4]).
We deal with sets and functions over the nonnegative integers. $\omega = \{0,1,2,\ldots\}$.
Let us define the function $\tau(x, y)$ as follows:

$$\tau(x,y) = \frac{1}{2}\left\{ x^2 + 2xy + y^2 + 3x + y \right\}.$$

The function $\tau(x, y)$ is a 1:1 recursive function from $\omega \times \omega$ onto ω. We shall use the symbol $< x, y >$ as an abbreviation for $\tau(x, y)$.
Let π_1 and π_2 denote the inverse functions $\pi_1(< x, y >) = x$ and $\pi_2(< x, y >) = y$.
$\varphi(x)\downarrow$ denotes that $\varphi(x)$ is defined, and $\varphi(x)\uparrow$ denotes that $\varphi(x)$ is undefined.
c_A denotes the characteristic function of A which is often identified with A and written simply as $A(x)$.

Definition 1. Let A be the nonempty finite set $\{x_1, \cdots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$. Then the integer $2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ is called a *canonical index of* A. If A is empty, the *canonical index* assigned to A is 0.

Let D_x be the finite set, the canonical index of which is x (see [4] p.70).

The definitions of tt - and btt - reducibilities are from [4].

Definition 2. (*i*) A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a *strong array* if there is a recursive function f such that $F_n = D_{f(n)}$.
(*ii*) An array is *disjoint* if its members are pairwise disjoint.
(*iii*) An infinite set B is *hyperimmune*, abbreviated *h-immune*, if there is no disjoint strong array $\{F_n\}_{n\in\omega}$ such that $F_n \cap B \neq \varnothing$ for all n.
(*iv*) An r.e. set A is *hypersimple*, abbreviated *h-simple,* if $\overline{A}$ is h-immune (see Soare [6], p. 80).

Definition 3. *(a)* The ordered pair $\ll x_1, \cdots, x_k >, \alpha >$, where $< x_1, \cdots, x_k >$ is a k-tuple of integers and α is a k-ary Boolean function ($k > 0$) is called a *truth-table condition (or tt-condition) of norm k*. The set $\{ x_1, \cdots, x_k \}$ is called an *associated set of the tt-condition.*
(b) The *tt-condition $\ll x_1, \cdots, x_k >, \alpha >$, is satisfied by* A if $\alpha(c_A(x_1), \cdots, c_A(x_k)) = 1$.

Notation. Each tt-condition is a finite object; clearly an effective coding can be chosen which maps all tt-conditions (of varying norm) onto ω.
Assume henceforth that such a particular coding has been chosen. When we speak of "tt-condition x", we shall mean the tt-condition with the code number x.
Code $\ll x_1, \cdots, x_k >, \alpha >$ denotes the code number of tt-condition $\ll x_1, \cdots, x_k >, \alpha >$ in this coding.

Definition 4. *(a) A is truth-table reducible to B (notation:* $A \leq_{tt} B$) if there is a recursive function f such that for all x [$x \in A \Leftrightarrow tt$-condition $f(x)$ is satisfied by B]. We also abbreviate *"truth-table reducibility"* as "*tt-reducibility*".

(b) A *is bounded truth-table reducible to* B *(notation:* $A \le_{btt} B$ *),* if ($\exists$recursive f) ($\exists m$)($\forall x$) [tt -condition $f(x)$ has norm $\le m$, and [$x \in A \Leftrightarrow f(x)$ is satisfied by B]].

We abbreviate "*bounded truth-table reducibility*" as "btt *-reducibility*" (see Rogers [4]).

2. PRELIMINARIES

Definition 5. Suppose $A \le_{tt} B$ and $(\forall x)$ [$x \in A \Leftrightarrow$ tt -condition $f(x)$ is satisfied by B] and $\varphi_n = f$. Then we say that $A \le_{tt} B$ *by* φ_n .

Definition 6. We say that $(A_0, A_1, \vartheta, \psi, e)$ is a *quasi-*btt *-mitotic splitting of* A if

a) (A_0, A_1) is a r.e. splitting of A and

b) $A \le_{btt} A_0$ by function ϑ with norm p_e (where $p_e = \pi_1(e)$) and

c) $A \le_{btt} A_1$ by function ψ with norm q_e (where $q_e = \pi_2(e)$).

Let us modify notations defined in (Lachlan [3]) with the purpose to adapt them to our theorem.

Notation. Let h be a recursive function from ω onto ω^5. Define ($Y_e, Z_e, \vartheta_e, \psi_e, j_e$) to be a quintuple ($W_{e_o}, W_{e_1}, \varphi_{e_2}, \varphi_{e_3}, e_4$), where $h(e) = (e_0, e_1, e_2, e_3, e_4)$.

Definition 7. If A is r.e. then we say that the *non-btt -mitotic condition of e order is satisfied for* A , if it is not the case that *(* $Y_e, Z_e, \vartheta_e, \psi_e, j_e$ *)* is a quasi-btt -mitotic splitting of A .

Notation. Let $x(e,s)$ be such a number that $\vartheta_{e,s}(x(e,s))\downarrow$ and $\psi_{e,s}(x(e,s))\downarrow$ (remind, that $\vartheta_e = \varphi_{e_2}$ and $\psi_e = \varphi_{e_3}$).

In this case

$as^2(e,s)$ denotes the associated set of tt -condition $\vartheta_e(x(e,s))$;

$as^3(e,s)$ denotes the associated set of tt -condition $\psi_e(x(e,s))$;

$as^*(e,s)$ denotes the set $as^2(e,s) \cup as^3(e,s)$.

If $\vartheta_{e,s}(x(e,s))\uparrow$ ($\psi_{e,s}(x(e,s))\uparrow$), then define $as^2(e,s) = \varnothing$ ($as^3(e,s) = \varnothing$).

If $(\theta_{e,s}(x(e,s))\uparrow \vee \psi_{e,s}(x(e,s))\uparrow)$, then define $as^*(e,s) = \varnothing$.

$assoc(e,s)$ denotes the set $\bigcup_{i=0}^{e} as^*(i,s)$.

Definition 8. $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt -threatening A through $x(e,s)$ *at stage* s , if all the following hold:

i) $Y_{e,s} \cap Z_{e,s} = \varnothing$,

ii) $(\forall y \le x(e,s))(\vartheta_{e,s}(y)\downarrow$ & $\psi_{e,s}(y)\downarrow)$ & $(\forall y \le x(e,s))$ [the norm of $\vartheta_e(y)$ is less or equal than p_{e_4} & the norm of $\psi_e(y)$ is less or equal than q_{e_4}], where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p_{e_4}$, $\pi_2(e_4) = q_{e_4}$.

iii) $x(e,s) \in A_s$ $\Leftrightarrow$ tt -condition $\vartheta_{e,s}(n)$ with norm p_{e_4} satisfied by $Y_{e,s}$) & $x(e,s) \in A_s$ $\Leftrightarrow$ tt -condition $\psi_{e,s}(n)$ with norm q_{e_4} satisfied by $Z_{e,s}$)],

iv) $A_s(m) = (Y_{e,s} \cup Z_{e,s})(m)$ for all $m \in as^*(x(e,s))$.

For the non-btt -mitotic condition the following proposition is true:

If $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt -threatening A through $x(e,s)$ at stage s, $x(e,s) \in A - A_s$ and for all $m \ne x(e,s)$ such that $m \in as^*(x(e,s))$ we have $A(m) = A_s(m)$, then the non-btt -mitotic condition of order e is satisfied for A .

This proposition is similar to Lemma 3 (about the nonmitotic condition) in (Lachlan [3]).

To satisfy the non-btt -mitotic condition of order e for A do the following. Have a number $x(e,s)$ (so called *follower*) in the complement of A ready to put into A if $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ happens to threaten A through x at some stage s and never put any other number belonging to $as^*(x(e,s))$ into A after stage $s+1$.

Definition 9. For any set $A \subseteq \omega$ and $x \in \omega$ define the x *-column of* A. $A^{(x)} = \{ <x,y> : <x,y> \in A \}$ (see Soare [5], p. 519).

Notation. $M_{y,x} = \omega^{(<y,x>)}$.

$M_e^0 = \bigcup_{i=0}^{\infty} M_{e,2i}$; $M_e^1 = \bigcup_{i=0}^{\infty} M_{e,2i+1}$.

$M^0 = \bigcup_{e=0}^{\infty} M_e^0$; $M^1 = \bigcup_{e=0}^{\infty} M_e^1$.

$\tilde{M}_{e,i} = M_{e,2i} \cup M_{e,2i+1}$; $M_e = \bigcup_{i=0}^{\infty} M_{e,i} = \bigcup_{i=0}^{\infty} \tilde{M}_{e,i}$.

Thus, $M^0 \cup M^1 = \omega$.

Let $a_0, a_1, \ldots, a_n, \ldots$ be the members of set A in increasing order. The integer a_i is denoted as $id(A)(i)$.

For any e, k define:

$M^*_{e,2k} = \{id(M_{e,2k})(1), id(M_{e,2k})(2), \dots,$
$id(M_{e,2k})(p_{e_4} + q_{e_4} + 1)\}$;
$M^*_{e,2k+1} = \{id(M_{e,2k+1})(0), id(M_{e,2k+1})(1), \dots,$
$id(M_{e,2k+1})(p_{e_4} + q_{e_4})\}$.

3. PROOF OF THE THEOREM

Let us prove the following theorem.

Theorem. *There exists a tt-mitotic hypersimple set, which is not btt-mitotic.*

Proof *(sketch).*
The theorem is proved using a finite injury priority argument. We construct a set A in stages s, $A = \bigcup_{s\in\omega} A_s$. The set A will be non-btt-mitotic and, withal, tt-mitotic and hypersimple.

We construct A to satisfy for all $e \in \omega$ the requirements:

R_e : The non-*btt*-mitotic condition of order e is satisfied for A.
P_e : $[[(\forall y)(\varphi_e(y)\downarrow) \& (u,v)((u \neq v) \Rightarrow$
$\Rightarrow D_{\varphi_e(u)} \cap D_{\varphi_e(v)} = \varnothing)] \Rightarrow (\exists y)(D_{\varphi_e(y)} \subseteq A)]$.

Note that if A is not *btt*-mitotic, then $\overline{A}$ is infinite.

Order the requirenments in the following priority ranking: $\tilde{R}_0, R_0, \tilde{R}_1, R_1, \tilde{R}_2, R_2, \dots$.

Definition 10. R_i *requires attention at stage* s if there exists such x that $(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt-threatening A through x at stage s and if it is not satisfied.

Construction

Stage $s = 0$: Let $A_0 = \varnothing$, $x(e,0) = id(M_{e,0})(0)$ for all e.

Stage $s+1$: Act on the highest priority requirement which requires attention, if such a requirement exists:

Case 1. Let R_e requires attention at stage s (through $x(e,s)$).
Let $x(e,s) \in M_{e,2k}$ for some k (that is $x(e,s) = id(M_{e,2k})(0)$).
Find z such, that $z \in M^*_{e,2k} \cup M^*_{e,2k+1}$ & $id(M^*_{e,2k+1})(z) \notin as^*(e,s)$ & $id(M^*_{e,2k})(z) \notin as^*(e,s)$.
Such an integer z exists certainly (because $(\forall s)\left[\sum_{i=0}^{e} \left|as^*(i,s)\right| \leq \sum_{i=0}^{e}(p_{i_4} + q_{i_4})\right]$, while $\left|M^*_{e,2k}\right| = \left|M^*_{e,2k+1}\right| = \sum_{i=0}^{e}(p_{i_4} + q_{i_4}) + 1$).

We choose the least such integer z_0. Set

$A_{s+1} = A_s \cup \{x(e,s)\} \cup \{id(M^*_{e,2k})(z_0)\} \cup \{id(M^*_{e,2k+1})(z_0)\}$.
Set $x(\hat{e}, s+1) = id(M_{\hat{e},2s})(0)$ for all $\hat{e} \geq e$.

Declare R_e *satisfied*, declare all lower R *unsatisfied*.

Case 2.
Notation. Define $l(e,s) = k$, where k is such that $x(e,s) = id(M_{e,2k})(0)$.
For all $y \in \omega$, if e,k,r are such that $y = id(M_{e,2k})(r) \vee y = id(M_{e,2k+1})(r)$, then define $od(y) = id(M_{e,2k+1})(r)$.
Note if y is such that $(\exists e,k,r)(y = id(M_{e,2k})(r))$ then $y = od(y)$.
If $(\exists m)[(\forall i \leq e)\varphi_{e,s}(m)\downarrow \& (\forall y,z)[(z \in D_{\varphi_e(m)}$ & $y \in \bigcup_{i=0}^{e} assoc(i,s) \cup \bigcup_{i=0}^{l(e,s)}(M^*_{e,2i} \cup M^*_{e,2i+1})) \Rightarrow$ $z > od(y)]]$, then let m_0 be the least of such m.
If P_e is not satisfied (at stage s) then for each z,k,y such that $z \in D_{\varphi_e(m_0)}$ and $(z = id(M_{e,2k})(y)$ or $z = id(M_{e,2k+1})(y))$ we set $id(M_{e,2k})(y) \in A_{s+1}$ and $id(M_{e,2k+1})(y) \in A_{s+1}$.

Note, that some elements, included into A in that way, could be included into A before the stage $s+1$.
Set $x(\hat{e}, s+1) = id(M_{\hat{e},2s})(0)$ for all $\hat{e} \geq e$.

Thus, P_e is satisfied, declare all lower R *unsatisfied*.

Verification

Lemma 1. $\lim_s x(e,s) = x(e)$ exists for all e.

Proof. By induction on e.
Suppose there exists a stage s_0 such that for all $\hat{e} < e$ $\lim_s x(\hat{e},s) = x(\hat{e})$ exists and is attained by s_0.

Then after stage s_0 only R_e and P_e can move $x(e,s)$. R_e and P_e, each taken separately, after s_0 acts at most once and is met. Therefore $(\exists \tilde{s} > s_0)(x(e,\tilde{s}) = \lim_s x(e,s))$.

Notation. Define $\tilde{A} = A \cap M^0$, $\tilde{\tilde{A}} = A \cap M^1$.

Lemma 2. $\tilde{A} \equiv_{tt} \tilde{\tilde{A}}$.

Let us prove that $\tilde{A} \equiv_{tt} \tilde{\tilde{A}}$ (where $\tilde{A} = A \cap M^0$, $\tilde{\tilde{A}} = A \cap M^1$). We must construct the function g_0 which *tt*-reduces $\tilde{A}$ to $\tilde{\tilde{A}}$ and the function g_1 which *tt*-reduces $\tilde{\tilde{A}}$ to $\tilde{A}$.
In this case there would exist recursive functions $\tilde{g}_0, \tilde{g}_1$ such that $A \leq_{tt} \tilde{\tilde{A}}$ by function $\tilde{g}_0$ and $A \leq_{tt} \tilde{A}$ by function $\tilde{g}_1$, because M^0, M^1 are recursive sets.

We will construct the functions g_0, g_1 according to the following considerations.

Construction of g_0: We shall indicate how to compute $g_0(x)$ for any x.

There are three cases to consider:

i) If $(\exists\, e)(\exists\, k)\,(\, x = id(M_{e,2k})(0))$, then define $g_0(x) = code <\!\!< id(M_{e,2k+1})\,(0), id(M_{e,2k+1})\,(1),\ldots,$ $id(M_{e,2k+1})\,(p+q) >\!, \alpha_1 >$

(where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_1(e_4) = p_{e_4}$, $\pi_2(e_4) = q_{e_4}$;

$$\alpha_1(x_0, x_1,\ldots, x_{p_{e_4}+q_{e_4}}) = \begin{cases} 0, \text{ if } x_0 = x_1 = \ldots = x_{p_{e_4}+q_{e_4}} = 0; \\ 1, \text{ otherwise.} \end{cases}).$$

ii) If $(\exists\, e)(\exists\, k > 0)\,(\, x \in M^*_{e,2k})$, then find z such that $x = id(M^*_{e,2k})\,(z)$.

Now define $g_0(x) = code <\!\!< id(M^*_{e,2k+1})\,(z), >\!, \alpha_2 >$, where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

iii) If $(\forall\, e)\,(\forall k)\,(x \notin \{ id(M_{e,2k})(0)\} \cup M^*_{e,2k})$, then find z such that $x = id(M_{e,2k})(z)$. Now define $g_0(x) = code <\!\!< id(M_{e,2k+1})(z) >\!, \alpha_2 >$, where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

Construction of g_1: We shall indicate how to compute $g_1(x)$ for any x.

There are two cases to consider:

i) If $(\exists\, e)(\exists k)(x \in M^*_{e,2k+1})$, then find z such that $x = id(M^*_{e,2k+1})\,(z)$. Now define $g_1(x) = code <\!\!< id(M^*_{e,2k})(z) >\!, \alpha_2 >$, where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

ii) If $(\forall\, e)(\forall k)\,(x \notin M^*_{e,2k+1})$, then find z such that $x = id(M^*_{e,2k+1})\,(z)$. Now define $g_1(x) = code <\!\!< id(M_{e,2k})(z) >\!, \alpha_2 >$, where $\alpha_2(x) = x$ for all $x \in \{0,1\}$.

The functions g_0, g_1 satisfy the abovementioned requirements.

Lemma 3. A is not btt-mitotic.

As mentioned above, $(\forall e)$ there exists a stage s_0 such that $(\forall s \ge s_0)\big(x(e,s_0) = x(e,s)\big)$.

For each e case *a)* or case *b)* takes place:

a) $(\neg\exists s \ge s_0)\,(\,(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt-threatening A through $x(e,s)$ at stage s). Therefore, the non-btt-mitotic condition of order e is satisfied for A.

b) $(\exists\, s \ge s_0)\,(\,(Y_e, Z_e, \vartheta_e, \psi_e, j_e)$ is btt-threatening A through $x(e,s)$ at stage s).

In this case the follower $x(e,s)$ will be put into A and non-btt-mitotic condition of order e will be satisfied.

Thus, set A is non-btt-mitotic.

Lemma 4. A is hypersimple.

For each $\hat{e}$ there exists s_0 such that $(\forall i \le \hat{e})(\forall s \ge s_0)\,(x(i,s_0) = x(i,s) = x(i))$.

So for each $\hat{e}$ there exists s_0 such that $(\forall i \le \hat{e})(\forall s \ge s_0)\,(l(i,s_0) = l(i,s) = l(i))$.

Therefore, for each $\hat{e}$ there exists s_0 such that $(\forall s \ge s_0)$ $(\bigcup_{i=0}^{l(\hat{e},s_0)} (M^*_{i,2i} \cup M^*_{i,2i+1}) = = \bigcup_{i=0}^{l(\hat{e},s)} (M^*_{i,2i} \cup M^*_{i,2i+1}) = \bigcup_{i=0}^{l(\hat{e})} (M^*_{i,2i} \cup M^*_{i,2i+1}))$.

For each $\hat{e}$ there exists s_0 such that $(\forall s \ge s_0)$ $assoc(\hat{e}, s_0) = assoc(\hat{e}, s) = assoc(\hat{e})$.

Also, for each $\hat{e}$ there exists s_0 such that $(\forall s \ge s_0)$ $\bigcup_{i=0}^{\hat{e}} assoc(i,s_0) = \bigcup_{i=0}^{\hat{e}} assoc(i,s) = \bigcup_{i=0}^{\hat{e}} assoc(i).$

Let φ_e be total function and $(\forall u,v)(u \ne v) \Rightarrow D_{\varphi_e(u)} \cap D_{\varphi_e(v)} = \varnothing$.

Then $(\exists m)(\forall y,z)[(z \in D_{\varphi_e(m)}\ \&\ y \in \bigcup_{i=0}^{e} assoc(i) \cup \bigcup_{i=0}^{l(e)} (M^*_{e,2i} \cup M^*_{e,2i+1}))\ \Rightarrow\ z > od(y)]]$. Therefore, there exist m_0,s_0 such that $(\forall z)(z \in D_{\varphi_{e,s_0}(m_0)} \Rightarrow\ z > od(y))$ for all y such that $y \in \bigcup_{i-0}^{e} assoc(i,s_0) \cup \bigcup_{i=0}^{l(e)} (M^*_{e,2i} \cup M^*_{e,2i+1})$ and Case 2 takes place at stage s_0+1. So $D_{\varphi_e(m_0)}$ is included in A at stage s_0+1. Thus P_e is met. ☐

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