On the Existence of the \( tt \)-Mitotic Hypersimple Set Which is not \( btt \)-Mitotic

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ABSTRACT

Let us adduce some definitions:
If a recursively enumerable (r.e.) set \( A \) is a disjoint union of two sets \( B \) and \( C \), then we say that \( B,C \) is an r.e. splitting of \( A \).
The r.e. set \( A \) is \( tt \)-mitotic (\( btt \)-mitotic) if there is an r.e. splitting \((B,C)\) of \( A \) such that the sets \( B \) and \( C \) both belong to the same \( tt \)-\( (btt) \)-degree of unsolvability, as the set \( A \).

In this paper the existence of \( tt \)-mitotic hypersimple set, which is not \( btt \)-mitotic is proved.

Keywords
Recursively enumerable (r.e.) set, hypersimple set, \( tt \)-reducibility, \( btt \)-reducibility.

1. INTRODUCTION

Notation. We shall use the notions and terminology introduced in (Soare [6]), (Downey and Stob [1]), (Rogers [4]).
We deal with sets and functions over the nonnegative integers. \( \omega = \{0,1,2,\ldots\} \).
Let us define the function \( \tau(x,y) \) as follows:
\[
\tau(x,y) = \frac{1}{2}\left\{x^2 + 2xy + y^2 + 3x + y \right\}.
\]
The function \( \tau(x,y) \) is a \( 1:1 \) recursive function from \( \omega \times \omega \) onto \( \omega \). We shall use the symbol \( \langle x,y \rangle \) as an abbreviation for \( \tau(x,y) \).
Let \( \pi_1 \) and \( \pi_2 \) denote the inverse functions \( \pi_1(<x,y>) = x \) and \( \pi_2(<x,y>) = y \).
\( \varphi(x) \downarrow \) denotes that \( \varphi(x) \) is defined, and \( \varphi(x) \uparrow \) denotes that \( \varphi(x) \) is undefined.
\( c_\alpha \) denotes the characteristic function of \( A \) which is often identified with \( A \) and written simply as \( A(x) \).

Definition 1. Let \( A \) be the nonempty finite set \( \{x_1,\ldots,x_s\} \), where \( x_1 < x_2 < \cdots < x_s \). Then the integer \( 2^{x_1} + 2^{x_2} + \cdots + 2^{x_s} \) is called a canonical index of \( A \). If \( A \) is empty, the canonical index assigned to \( A \) is \( 0 \).

Let \( D \) be the finite set, the canonical index of which is \( x \) (see [4] p.70).
The definitions of \( tt \)- and \( btt \)-reducibilities are from [4].

Definition 2. (i) A sequence \( \{F_n\}_{n \in \omega} \) of finite sets is a strong array if there is a recursive function \( f \) such that \( F_n = D_{f(n)} \).
(ii) An array is disjoint if its members are pairwise disjoint.
(iii) An infinite set \( B \) is hyperimmune, abbreviated \( h \)-immune, if there is no disjoint strong array \( \{F_n\}_{n \in \omega} \) such that \( F_n \cap B \neq \emptyset \) for all \( n \).
(iv) An r.e. set \( A \) is hypersimple, abbreviated \( h \)-simple, if \( \overline{A} \) is \( h \)-immune (see Soare [6], p. 80).

Definition 3. (a) The ordered pair \( \langle x_1,\ldots,x_k,\alpha \rangle \) is a \( k \)-tuple of integers and \( \alpha \) is a \( k \)-ary Boolean function \((k > 0)\) is called a truth-table condition (or \( tt \)-condition) of norm \( k \). The set \( \{x_1,\ldots,x_k\} \) is called an associated set of the \( tt \)-condition.
(b) The \( tt \)-condition \( \langle x_1,\ldots,x_k,\alpha \rangle \) is satisfied by \( A \) if \( \alpha(c_\alpha(x_1),\ldots,c_\alpha(x_k)) = 1 \).

Notation. Each \( tt \)-condition is a finite object; clearly an effective coding can be chosen which maps all \( tt \)-conditions (of varying norm) onto \( \omega \).
Assume henceforth that such a particular coding has been chosen. When we speak of “\( tt \)-condition \( x \)”, we shall mean the \( tt \)-condition with the code number \( x \).

Code \( \langle x_1,\ldots,x_k,\alpha \rangle \) denotes the code number of \( tt \)-condition \( \langle x_1,\ldots,x_k,\alpha \rangle \) in this coding.

Definition 4. (a) \( A \) is truth-table reducible to \( B \) (notation: \( A \leq_{tt} B \)) if there is a recursive function \( f \) such that for all \( x \) \( \{x \in A \iff \text{\( tt \)-condition \( f(x) \) is satisfied by \( B \}\} \).
We also abbreviate “truth-table reducibility” as “\( tt \)-reducibility.”
(b) A is bounded truth-table reducible to B (notation: $A \leq_{tt} B$), if (1) recursive $f$ (2) $\forall x \in X \ (tt\text{-condition } f(x)\text{ has norm } \leq m)$, and \( \{ x \in A \iff f(x) \text{ is satisfied by } B \} \).

We abbreviate “bounded truth-table reducibility” as “btt-reducibility” (see Rogers [4]).

2. PRELIMINARIES

Definition 5. Suppose $A \leq_{tt} B$ and $(\forall x \in A \Rightarrow \forall x \in X \ (tt\text{-condition } f(x)\text{ is satisfied by } B))$ and $\varphi_n = f$. Then we say that $A \leq_{tt} B$ by $\varphi_n$.

Definition 6. We say that $(A_i, A_{i+1}, \vartheta, \psi, e)$ is a quasi-btt-splitting of $A$ if

a) $(A_i, A_{i+1})$ is a r.e. splitting of $A$ and

b) $A \leq_{tt} A_i$ by function $\vartheta$ with norm $p_e$ (where $p_e = \pi_1(e)$) and

c) $A \geq_{tt} A_i$ by function $\psi$ with norm $q_e$ (where $q_e = \pi_2(e)$).

Let us modify notations defined in (Lachlan [3]) with the purpose to adapt them to our theorem.

Notation. Let $h$ be a recursive function from $\omega$ onto $\omega^5$. Define $(Y_i, Z_i, \vartheta_i, \psi_i, e_i)$ to be a quintuple $(W_i, W_i, \varphi_i, \varphi_i, e_i)$, where $h(e) = (e_0, e_1, e_2, e_3, e_4)$.

Definition 7. If $A$ is r.e. then we say that the non-btt-splitting of $e$ order is satisfied for $A$, if it is not the case that $(Y_i, Z_i, \vartheta_i, \psi_i, e_i)$ is a quasi-btt-splitting of $A$.

Notation. Let $x(e, s)$ be such a number that $\vartheta_i(x(e, s)) \downarrow$ and $\psi_i(x(e, s)) \downarrow$ (remind, that $\vartheta_i = \varphi_i$ and $\psi_i = \varphi_i$).

In this case $as^2(e, s) = \{ \vartheta_i(x(e, s)) \}$ denotes the associated set of $tt$-condition $\vartheta_i(x(e, s))$; $as^3(e, s) = \{ \psi_i(x(e, s)) \}$ denotes the associated set of $tt$-condition $\psi_i(x(e, s))$; $as^4(e, s) = as^2(e, s) \cup as^3(e, s)$.

If $\vartheta_i(x(e, s))^\uparrow \psi_i(x(e, s))^\uparrow$, then define $as^5(e, s) = \{ x(e, s) \}$.

If $\vartheta_i(x(e, s))^\uparrow \lor \psi_i(x(e, s))^\uparrow$, then define $as^6(e, s) = \{ x(e, s) \}$.

assoc(e, s) denotes the set $\bigcup_{i=0}^{e} as^7(i, s)$.

Definition 8. $(Y_i, Z_i, \vartheta_i, \psi_i, e_i)$ is btt-threatening $A$ through $x(e, s)$ at stage $s$, if all the following hold:

i) $Y_i \cap Z_i = \emptyset$;

ii) $(\forall y \leq x(e, s))(\exists i)(\vartheta_i(y) \downarrow \land \psi_i(y) \downarrow$ and $\forall y \leq x(e, s)) \ (\text{the norm of } \vartheta_i(y) \text{ is less or equal than } p_e \land \text{ the norm of } \psi_i(y) \text{ is less or equal than } q_e)$, where $h(e) = (e_0, e_1, e_2, e_3, e_4)$, $\pi_i(e_i) = p_{e_i}$, $\pi_2(e_i) = q_{e_i}$.

iii) $x(e, s) \in A_i \iff tt\text{-condition } \vartheta_i(x(e, s)) \downarrow$.

For the non-btt-splitting the following proposition is true:

If $(Y_i, Z_i, \vartheta_i, \psi_i, e_i)$ is btt-threatening $A$ through $x(e, s)$ at stage $s$, $x(e, s) \in A_i \implies A_i \not\subseteq \emptyset$, and for all $m \neq x(e, s)$ such that $m \in as^5(x(e, s))$ we have $A(m) = A(m)$,

then the non-btt-splitting of $e$ order is satisfied for $A$.

This proposition is similar to Lemma 3 (about the nonmitotic condition) in (Lachlan [3]).

To satisfy the non-btt-splitting order $e$ for $A$ do the following. Have a number $x(e, s)$ (so-called follower) in the complement of $A$ ready to put into $A$ if $(Y_i, Z_i, \vartheta_i, \psi_i, e_i)$ happens to threaten $A$ through $x$ at some stage $s$ and never put any other number belonging to $as^5(x(e, s))$ into $A$ after stage $s+1$.

Definition 9. For any set $A \subseteq \omega$ and $x \in \omega$ define the $x$-column of $A$. $A^{(x)} = \{ y \mid y > x, y > A \}$ (see Soare [5], p. 519).

Notation. $M_{y,x} = \omega^{<\langle y,x \rangle}$. $M_{+}^{e} = \bigcup_{i=0}^{e} M_{e,2i};$ $M_{+}^{e} = \bigcup_{i=0}^{e} M_{e,2i}$. $M_{-}^{e} = \bigcup_{i=0}^{e} M_{e,2i}.$ $M_{+}^{e} = \bigcup_{i=0}^{e} M_{e,2i};$ $M_{+}^{e} = \bigcup_{i=0}^{e} M_{e,2i}$. $M_{-}^{e} = \bigcup_{i=0}^{e} M_{e,2i}.$

Thus, $M_{+}^{e} \cup M_{-}^{e} = \omega$.

Let $a_0, a_1, \ldots, a_n, \ldots$ be the members of set $A$ in increasing order. The integer $a_i$ is denoted as $i(d(A)(i))$.

For any $e, k$ define:
\begin{align*}
M_{e,2k}^* = \{id(M_{e,2k})(1), id(M_{e,2k})(2), \ldots, \\
id(M_{e,2k})(p_e + q_e + 1)\} ;
\end{align*}
\begin{align*}
M_{e,2k+1}^* = \{id(M_{e,2k+1})(0), id(M_{e,2k+1})(1), \ldots, \\
id(M_{e,2k+1})(p_e + q_e)\} .
\end{align*}

3. PROOF OF THE THEOREM
Let us prove the following theorem.

**Theorem.** There exists a \(\text{tbt}\)-mitotic hypersimple set, which is not \(\text{btt}\)-mitotic.

**Proof (sketch).** The theorem is proved using a finite injury priority argument. We construct a set \(A\) in stages \(s\), \(A = \bigcup_{s \in \omega} A_s\). The set \(A\) will be non-\(\text{btt}\)-mitotic and, withal, \(\text{tbt}\)-mitotic and hypersimple.

We construct \(A\) to satisfy for all \(e \in \omega\) the requirements:

\(R_e:\) The non-\(\text{btt}\)-mitotic condition of order \(e\) is satisfied for \(A\).

\(P_e:\) \(\exists (\forall y)(\varphi_y(y) \land (u,v)(u \neq v) \Rightarrow D_{G(e)}(y) \land D_{G(e)}(v) = \emptyset) \Rightarrow (\exists y)(D_{G(e)}(y) \subseteq A)\).

Note that if \(A\) is not \(\text{btt}\)-mitotic, then \(\overline{A}\) is infinite.

Order the requirements in the following priority ranking:

\(R_{s_0}, R_{s_1}, R_{s_2}, R_{s_3}, \ldots\).

**Definition 10.** \(R_e\) requires attention at stage \(s\) if there exists some \(x\) that \((Y_e, Z_e, \varnothing_e, \varphi_e, j_e)\) is \(\text{btt}\)-threatening \(A\) through \(x\) at stage \(s\) and if it is not satisfied.

**Construction**

**Stage \(s = 0\):** Let \(A_0 = \emptyset\), \(x(e,0) = id(M_{e,0})(0)\) for all \(e\).

**Stage \(s + 1\):** Act on the highest priority requirement which requires attention, if such a requirement exists:

**Case 1.** Let \(R_e\) requires attention at stage \(s\) (through \(x(e,s)\)).

Let \(x(e,s) \in M_{e,2k}^*\) for some \(k\) (that is \(x(e,s) = id(M_{e,2k}^*)(0)\)).

Find \(z\) such that \(z \in M_{e,2k}^* \cup M_{e,2k+1}^* \land id(M_{e,2k+1})(z) \not\in as^s(e,s) \land id(M_{e,2k}^*)(z) \not\in as^s(e,s)\).

Such an integer \(z\) exists certainly (because \((\forall s)[\sum_{i=0}^{s}\lceil as^i(e,s) \rceil \leq \sum_{i=0}^{s}(p_e + q_e)]\), while \(M_{e,2k}^* = \sum_{i=0}^{s}(p_e + q_e) + 1\).

We choose the least such integer \(z_{s}^*\). Set \(A_{s+1} = A_s \cup \{x(e,s)\} \cup \{id(M_{e,2k}^*)(z_{s})\} \cup \{id(M_{e,2k+1}^*)(z_{s})\}\).

Set \(x(e,s + 1) = id(M_{e,2k}^*)(0)\) for all \(\hat{e}, s \geq e\).

Declare \(R_e\) satisfied, declare all lower \(R\) unsatisfied.

**Case 2.**

**Notation.** Define \(l(e,s) = k\), where \(k\) is such that \(x(e,s) = id(M_{e,2k}^*)(0)\).

For all \(y \in \omega\), if \(e, k, r\) are such that \(y = id(M_{e,2k}^*)(r) \lor y = id(M_{e,2k+1}^*)(r)\), then define \(od(y) = id(M_{e,2k+1}^*)(r)\).

Note if \(y\) is such that \((\exists e, k, r)(y = id(M_{e,2k}^*)(r))\) then \(y = od(y)\).

If \((\exists m)((\forall i \leq e)\varphi_{e,i}(m) \land (\forall y, z)(z \in D_{\varphi_i(m)} \land y \in D_{\varphi_i(m)} \land z > od(y)))\), then let \(m_0\) be the least of such \(m\).

If \(P_r\) is not satisfied (at stage \(s\)) then for each \(z, k, y\) such that \(z \in D_{\varphi_i(m)}\) and \((z = id(M_{e,2k}^*)(y) \lor z = id(M_{e,2k+1}^*)(y))\) we set \(od(M_{e,2k}^*)(y) \in A_{s+1} \land id(M_{e,2k+1}^*)(y) \in A_{s+1}\).

Note, that some elements, included into \(A\) in that way, could be included into \(A\) before the stage \(s + 1\).

Set \(x(e,s + 1) = id(M_{e,2k}^*)(0)\) for all \(\hat{e}, s \geq e\).

Thus, \(P_r\) is satisfied, declare all lower \(R\) unsatisfied.

**Verification**

**Lemma 1.** \(\lim_{e} x(e,s) = x(e)\) exists for all \(e\).

**Proof.** By induction on \(e\).

Suppose there exists a stage \(s_0\) such that for all \(\hat{e}, s \leq e\) \(\lim_{e} x(e, s) = x(\hat{e})\) exists and is attained by \(s_0\).

Then after stage \(s_0\) only \(R_e\) and \(P_r\) can move \(x(e,s)\). \(R_e\) and \(P_r\), each taken separately, after \(s_0\) acts at most once and is met. Therefore \((\exists s > s_0)(x(e,s) = \lim_{e} x(e,s))\).

**Notation.** Define \(\tilde{A} = A \cap M^0. \tilde{A} = A \cap M^1\).

**Lemma 2.** \(\tilde{A} \equiv_\pi \tilde{A}\).

Let us prove that \(\tilde{A} \equiv_\pi \tilde{A}\) (where \(\tilde{A} = A \cap M^0\), \(\tilde{A} = A \cap M^1\)). We must construct the function \(g_0\) which \(u\)-reduces \(A\) to \(\tilde{A}\) and the function \(g_1\) which \(u\)-reduces \(A\) to \(\tilde{A}\).

In this case there would exist recursive functions \(\tilde{g}_0\) and \(\tilde{g}_1\), because \(M^0, M^1\) are recursive sets.
We will construct the functions $g_0$, $g_1$ according to the following considerations.

Construction of $g_0$: We shall indicate how to compute $g_0(x)$ for any $x$.

There are three cases to consider:

i) If $(\exists e)(\exists k)(x = id(M_{e,2k}^*)(0))$, then define $g_0(x) = \text{code} \ll \text{id}(M_{e,2k}^*)(0), \text{id}(M_{e,2k}^*)(1), \ldots,$ $\text{id}(M_{e,2k}^*)(p + q) \gg, \alpha_2 >$

(where $h(e) = (e_0, e_1, e_2, e_3, e_4)_e$, $\pi_1(e_4) = p_{e_4}$,

\[ \pi_2(e_4) = q_{e_4} \];

\[ \alpha(x_0, x_1, \ldots, x_{p_{e_4}} + q_{e_4}) = \begin{cases} 0, & \text{if } x_0 = x_1 = \ldots = x_{p_{e_4}} + q_{e_4} = 0; \\ 1, & \text{otherwise}. \end{cases} \]

ii) If $(\exists e)(\exists k > 0)(x \in M_{e,2k}^*)$, then find $z$ such that $x = id(M_{e,2k}^*)(z)$. Now define $g_0(x) = \text{code} \ll \text{id}(M_{e,2k}^*)(z), \alpha_2 >$

where $\alpha_2(x) = x$ for all $x \in \{0, 1\}$.

iii) If $(\forall e)(\forall k)(x \in \{\text{id}(M_{e,2k}^*)(0)\} \cup M_{e,2k}^*)$, then find $z$ such that $x = id(M_{e,2k}^*)(z)$. Now define $g_0(x) = \text{code} \ll \text{id}(M_{e,2k}^*)(z), \alpha_2 >$

where $\alpha_2(x) = x$ for all $x \in \{0, 1\}$.

Construction of $g_1$: We shall indicate how to compute $g_1(x)$ for any $x$.

There are two cases to consider:

i) If $(\exists e)(\exists k)(x \in M_{e,2k}^*)$, then find $z$ such that $x = id(M_{e,2k}^*)(z)$. Now define $g_1(x) = \text{code} \ll \text{id}(M_{e,2k}^*)(z), \alpha_2 >$

where $\alpha_2(x) = x$ for all $x \in \{0, 1\}$.

ii) If $(\forall e)(\forall k)(x \in M_{e,2k}^*)$, then find $z$ such that $x = id(M_{e,2k}^*)(z)$. Now define $g_1(x) = \text{code} \ll \text{id}(M_{e,2k}^*)(z), \alpha_2 >$

where $\alpha_2(x) = x$ for all $x \in \{0, 1\}$.

The functions $g_0$, $g_1$ satisfy the abovementioned requirements.

**Lemma 3.** $A$ is not \textit{btt} -mitotic.

As mentioned above, $(\forall e)$ there exists a stage $s_0$ such that $(\forall s \geq s_0)(x(e, s), x(e, s)) = x(e, s).

For each $e$ case a) or case b) takes place:

a) $(\neg \exists s \geq s_0)(x(e, s), x(e, s))$ is \textit{btt} -threatening $A$ through $x(e, s)$ at stage $s$. Therefore, the non-\textit{btt} -mitotic condition of order $e$ is satisfied for $A$.

b) $(\exists s \geq s_0)$ $(Y, Z, \vartheta, \psi, j)$ is \textit{btt} -threatening $A$ through $x(e, s)$ at stage $s$.

In this case the follower $x(e, s)$ will be put into $A$ and non-\textit{btt} -mitotic condition of order $e$ will be satisfied.

Thus, set $A$ is non-\textit{btt} -mitotic.

**Lemma 4.** $A$ is hypersimple.

For each $\hat{e}$ there exists $s_0$ such that $(\forall i \leq \hat{e})(\forall s \geq s_0)(x(i, s) = x(i))$.

So for each $\hat{e}$ there exists $s_0$ such that

$(\forall i \leq \hat{e})(\forall s \geq s_0)(l(i, s) = l(i))$.

Therefore, for each $\hat{e}$ there exists $s_0$ such that $(\forall s \geq s_0)$

$\text{assoc}(\hat{e}, s_0) = \text{assoc}(\hat{e}, s) = \text{assoc}(\hat{e})$.

Also, for each $\hat{e}$ there exists $s_0$ such that $(\forall s \geq s_0)$

$\text{assoc}(\hat{e}, s_0) = \text{assoc}(\hat{e}, s) = \text{assoc}(\hat{e})$.

Let $\varphi_e$ be total function and $(\forall u, v)(u \neq v) \Rightarrow D_{\varphi_e(u)} \cap D_{\varphi_e(v)} = \emptyset$.

Then $(\exists m)(\forall y, z)(z \in D_{\varphi_m(y)} \land y \in \bigcup_{i=0}^{l(e)} \text{assoc}(i)) \Rightarrow z > od(y)$]. Therefore, there exist $m_{0, s_0}$ such that $(\forall z)(z \in D_{\varphi_{m_{0, s_0}}(y)} \Rightarrow z > od(y))$ for all $y$ such that $y \in \bigcup_{i=0}^{l(e)} \text{assoc}(i, s_0) \cup \bigcup_{i=0}^{l(e)} M_{0, s_0}$.

And Case 2 takes place at stage $s_0 + 1$. Thus $P_e$ is met.

**REFERENCES**


