On Belousov Quasigroups *

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ABSTRACT
In this paper we characterize the variety of Belousov quasigroups by bigroups and finite fields.

Keywords
Quasigroup, bigroup, finite field, Mikado variety, word problem, simple algebra, cellular automaton.

The quasigroup $Q(\circ)$ is called a Belousov quasigroup,if the identities

$$x \circ (x \circ y) = y \circ x,$$

$$(x \circ y) \circ y = x,$$

$$x \circ (y \circ x) = (y \circ x) \circ y$$

are valid. A non-trivial Belousov quasigroup is not a Stein quasigroup and not commutative.

The set $O_p^{(2)}Q$ of all binary operations on the set Q is a monoid under the following operations ([3, 4, 5]):

$$f \cdot g(x,\, y) = f(x,\, g(x,\, y)), \tag{1}$$

$$f \circ g(x,\, y) = f(g(x,\, y),\, y). \tag{2}$$

Theorem 1. *If $Q(A)$ is a non-trivial Belousov quasigroup, then it is idempotent and $A \cdot A = A^*$, $A \cdot A^* = A \circ A^*$, $A \circ A = \delta_2^1$, $A^* \cdot A^* = \delta_2^2, A^* \circ A^* = A$. So if $Q(A)$ is a non-trivial Belousov quasigroup, then the set $\{\delta_2^1, \delta_2^2, A, A^*, A \cdot A^* = A \circ A^*\}$ is a bigroup of operations (on the set Q), where $A^*(x,y) = A(y,x)$ for every $x, y \in Q$.*

Theorem 2. *In every Belousov quasigroup $Q(\circ)$ the identities $(x \circ y) \circ (y \circ x) = y, (x \circ y) \circ (x \circ (y \circ x)) = y \circ x, (y \circ x) \circ (x \circ (y \circ x)) = x \circ y$ are valid. In a non-trivial Belousov quasigroup $Q(\circ)$, for any $a \neq b$ in Q the set $\{a, b, a \circ b, b \circ a, a \circ (b \circ a)\}$ is a five-element subquasigroup, which is isomorphic to the five-element quasigroup with the following multiplication table:*

	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

If we take such subquasigroups as blocks, we obtain a block design on the set Q.

Theorem 3. *If $Q(\circ)$ is a non trivial Belousov quasigroup, then for every $u, v \in Q, u \neq v$ there exists a five-element field $H_{u,v}(+, \cdot), H_{u,v} \subseteq Q$ such that $u, v \in H_{u,v}$ and for every $x, y \in H_{u,v}$:*

$$x \circ y = (y - x)a + x, a \in H.$$

It follows from the last Theorem 3 (or Theorem 2) that the non-trivial Belousov quasigroup has at least five elements. The variety of Belousov quasigroups is called a Belousov variety, which is a subvariety of the Mikado variety ([1]). Hence, the Belousov variety has a solvable word problem and is congruence-permutable. Every Belousov quasigroup of prime order is a simple algebra.

The applications of similar quasigroups in cellular automata see in [2].

To solution of the following problem is open.

To which loops are Belousov quasigroups isotopic?

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