Two Generalized Lower Bounds for the Circumference

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ABSTRACT

Two lower bounds for the circumference (the length of a longest cycle C in a graph G) are presented in terms of a longest path (a longest cycle) in G-C and the average of the first i smallest degrees in G. As immediate corollaries, we obtain the original lower bounds for the circumference in terms of G-C structures and the minimum degree of G (Zh.G. Nikoghosyan, "Advanced Lower Bounds for the Circumference", Graphs and Combinatorics 29, pp. 1531-1541, 2013).

Keywords

Circumference.

1. INTRODUCTION

Let c be the circumference - the length of a longest cycle C in a graph G. Denote by $\hat{p}(C)$ and $\hat{c}(C)$ the lengths of a longest path and a longest cycle in G-C, respectively. Further, let $\hat{p}(G)=\min_C \hat{p}(C)$, $\hat{c}(G)=\min_C \hat{c}(C)$ and let μ_i be the average of the first i smallest degrees in G. For the simplicity, we will write $\hat{p}$ and $\hat{c}$ instead of $\hat{p}(G)$ and $\hat{c}(G)$.

In this paper we present the following two results.

Theorem 1. Let C be any cycle in a graph G and Q any path of length q in G-C. Then either there is a path in G-C longer than Q or there is a cycle in G longer than C or $c \ge (q+2)(\mu_q - q)$.

Theorem 2. Let C be any cycle in a graph G and H any cycle of length h in G-C. Then either there is a cycle in G-C longer than H or there is a cycle in G longer than C or $c \ge (\mathrm{h}+1)(\mu_h - \mathrm{h}+1)$.

As corollaries we give the following two main results in terms of $\hat{p}, \hat{c}, \mu_{\hat{p}}, \mu_{\hat{c}}$.

Theorem 3. $c \ge (\hat{p}+2)(\mu_{\hat{p}} - \hat{p})$.

Theorem 4. $c \ge (\hat{c}+1)(\mu_{\hat{c}} - \hat{c}+1)$.

Let δ be the minimum degree in G. Observing that $\mu_{\hat{p}} \ge \delta$ and $\mu_{\hat{c}} \ge \delta$, we obtain the original lower bounds [2] as immediate corollaries in terms of $\hat{p}(C)$, $\hat{c}(C)$ and δ.

Theorem A1 [2]. For each longest cycle C in G,

$$c \ge (\hat{p}(C)+2)(\delta - \hat{p}(C)).$$

Theorem A2 [2]. For each longest cycle C in G,

$$c \ge (\hat{c}(C)+1)(\delta - \hat{c}(C)+1).$$

2. DEFINITIONS

We use Bondy and Murty [1] for terminology and notation not defined here, and consider only finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by V(G) or just V; the set of edges by E(G) or just E. For a subgraph H of G we also use G-H short for G-V(H), and |H| short for |V(H)|.

Paths and cycles in G can be considered as connected subgraphs of G, having a maximum degree 0,1 or 2. The length of a path P and of a cycle Q, denoted by *l*(P) and *l*(Q), is |V(P)|-1 and |V(Q)|, respectively. We denote *l*(P)=-1 and *l*(Q)=0 if and only if V(P)=V(Q)=∅. A graph is said to be Hamiltonian if its longest cycle passes through all of its vertices. The vertices and edges in G can be interpreted as cycles of lengths 1 and 2, respectively.

An (x,y)-path is a path with end vertices x and y. Given an (x,y)-path L of G we denote by $\vec{L}$ the path L with an orientation from x to y. If u,v∈V(L) then $u\vec{L}v$ denotes the consecutive vertices on $\vec{L}$ from u to v in the direction specified by $\vec{L}$. The same vertices, in reverse order, are given by $v\overleftarrow{L}u$. For $\vec{L}=x\vec{L}y$ and $u \in V(L)$, let $u^+(\vec{L})$ (or just u^+) denote the successor of u (u≠y) on $\vec{L}$, and u^- denote its predecessor (u≠x). If A⊆V(L)-y and B⊆V(L)-x then we denote $A^+=\{v^+|v\in A\}$ and $B^-=\{v^-|v\in B\}$. Similar notation is used for the cycles. If Q is a cycle and u∈V(Q), then $u\vec{Q}u=u$. For v∈V, put N(v)={u∈V| uv∈E}, d(v)=|N(v)| and δ=min{d(u)|u∈V}.

3. SPECIAL DEFINITIONS

For the remainder of this section, let a subgraph F of a graph G and a path (or a cycle) $\vec{M}$ in G-F be fixed.

Definition 1. $(*\,i)$-minimality, $(*\,i)$-maximality.

We use the notions of $(*\,i)$-minimality and $(*\,i)$-maximaliy defined with respect to certain operations for i=1,2,…,10. They will be described in detailed currently.

Definition 2. *MF-extension;* $\vec{\boldsymbol{T}}(\boldsymbol{u})$*;* u̇; $\ddot{u}$.

For each u∈V(M), let $\vec{T}(u)=u\vec{T}(u)\ddot{u}$ be a path in G, having only u in common with V(M). If V(T(u))∩V(T(v))=∅ and $V(T(u)) \subseteq V(G-F)$ for all distinct vertices u,v∈V(M) then the forest T, defined by {T(u)|u∈V(M)}, is said to be MF-extension. If $u \ne \ddot{u}$ for some $u \in V(M)$, then we use $\dot{u}$ to denote $u^+\left(\vec{T}(u)\right)$.

Definition 3. Φ_u; φ_u; Ψ_u; ψ_u.

Let T be an MF-extension. For each u∈V(M), put

$$\Phi_u = N(\ddot{u}) \cap V(T),\ \varphi_u = |\Phi_u|,$$

$$\Psi_u = N(\ddot{u}) \cap V(F),\ \psi_u = |\Psi_u|.$$

Definition 4. U_0; $\overline{U}_0$; U_1; U^*.

For T an MF-extension, put

$U_0=\{u \in V(M)|u=\ddot{u}\}$; $\overline{U}_0 = V(M)-U_0$,

$U^*=\{u\in \overline{U}_0 | \Phi_u \subseteq V(T(u))\}$; $U_1 = V(M)-(U_0 \cup U^*)$.

Definition 5. Maximal MF-extension.

An MF-extension T is said to be maximal if it is extremal with respect to the following operation:

- if there exists an edge $\ddot{u}z$ such that $u \in V(M)$ and $z \notin V(T) \cup V(F)$, then replacing T(u) by $uT(u)\ddot{u}z$, we obtain a new MF-extension T' with $|V(T')| > |V(T)|$.

Definition 6. *(U_0)-minimal and (U_0,U^*)-minimal MF-extensions.*

An MF-extension T is said to be (U_0)-minimal, if it is chosen such that U_0 is $(*6)$-minimal (see the proof of Theorem 1). A (U_0)-minimal MF-extension T is said to be (U_0,U^*)-minimal if it is chosen such that U^* is $(*10)$ –minimal (see the proof of Theorem 2).

Definition 7. B_u; B_u^*; b_u; b_u^*.

For T an MF-extension and u∈V(M), let $B_u = \{v \in U_0 | v\dot{u} \in E\}$ and $b_u = |B_u|$. By the definition, $B_u = \emptyset$ for each u∈U_0 . Furthermore, for each u∈U_0 , let $B_u^* = \{v \in \overline{U}_0 | u\dot{v} \in E\}$ and $|B_u^*| = b_u^*$.

4. PRELIMINARIES

The proofs of the following lemmas can be find in [2].

Lemma 1. Let C be a cycle in a graph G and P a path in G-C. Let $\vec{P}_0, \dots, \vec{P}_p$ be pairwise disjoint paths in G-C with $\vec{P}_i = v_i \vec{P}_i w_i$ (i=0,1,…,p), having only v_0,…,v_p in common with P. Then either there is a cycle in G longer than C or

$$|\boldsymbol{C}| \geq \sum_{i=0}^{p} |\boldsymbol{Z}_i| + \left|\bigcup_{i=0}^{p} \boldsymbol{Z}_i\right|,$$

where $Z_i = N(w_i) \cap V(C) \quad (i = 0,1, \dots, p)$.

Lemma 2. Let F be a subgraph of a graph G and R a longest cycle in G-F with a (U_0)-minimal RF-extension T. Then either there is a cycle longer than R or $l(R) \geq \varphi_u + b_u + 1$ for each u∈U_1.

Lemma 3. Let F be a subgraph of a graph G and P a path in G-F with a (U_0)-minimal PF-extension T. Then either there is a path longer than P or $l(P) \geq \varphi_u + b_u$ for each $u \in U_1 \cup U^*$.

5. PROOFS

We present the proof of Theorem 1.

Proof of Theorem 1. Let $Q = u_0 \dots u_q$ be a path in G-C with a (U_0)- minimal QC-extension T. Assume without loss of generality that C is $(*1-4)$-extremal and Q is $(*7-9)$ -extremal. Since G is non-Hamiltonian, we have $q \geq 0$.

Claim 1. If $u \in U_0$ and $v \in \overline{U}_0$ then $\Phi_u \cap V(T(v)) \subseteq \{v, \dot{v}\}$.

Proof. Suppose otherwise. Let $z \in V(T(v)) - \{v, \dot{v}\}$. Then, replacing T(u) and T(v) by $uz\vec{T}(v)\dot{v}$ and $v\vec{T}(v)z^-$, respectively, we can form (denote this operation by $(*6)$) a new QC-extension, contradicting the (U_0)- minimality of T.

Claim 2. If $u \in U_0$, then $\varphi_u \leq q + b_u^*$.

Proof. The proof follows immediately from Definitions 3, 7 and Claim 1.

Claim 3. If $u \in \overline{U}_0$, then $\varphi_u \leq q - b_u$.

Proof. Using Lemma 3 with the fact that Q is $(*7-9)$-extremal, we obtain $q \geq \varphi_u + b_u$ for each $u \in \overline{U}_0$, and the result follows.

Observing that $\sum_{u \in U_0} b_u^* = \sum_{u \in \overline{U}_0} b_u$ (by the definition) and using Claims 2 and 3, we obtain

$$\sum_{i=0}^{q} \varphi_{u_i} \leq q(q+1) + \sum_{u \in U_0} b_u^* - \sum_{u \in \overline{U}_0} b_u = q(q+1).$$

Suppose first that $\varphi_{u_i} + \psi_{u_i} \neq d(\ddot{u}_i)$ for some $i \in \overline{0,q}$. Then there exist an edge $\ddot{u}z$ such that $z \notin V(T) \cup V(C)$. Adding $\ddot{u}z$ to T we obtain a new QC-extension, contradicting the maximality of T (Definition 5). Now let $\varphi_{u_i} + \psi_{u_i} = d(\ddot{u}_i)$ $(i = 0, \dots, q)$. Then

$$\sum_{i=0}^{q} \psi_{u_i} = \sum_{i=0}^{q} d(\ddot{u}_i) - \sum_{i=0}^{q} \varphi_{u_i} \geq \sum_{i=0}^{q} d(\ddot{u}_i) - q(q+1).$$

It follows, in particular, that

$$max_i\{\psi_{u_i}\} \geq \frac{1}{q+1} \sum_{i=0}^{q} \psi_{u_i} \geq \frac{1}{q+1} \sum_{i=0}^{q} d(\ddot{u}_i) - q.$$

By Lemma 1,

$$c \geq \sum_{i=0}^{q} \psi_{u_i} + max_i\{\psi_{u_i}\} \geq (q+2)\left(\frac{1}{q+1} \sum_{i=0}^{q} d(\ddot{u}_i) - q\right)$$
$$\geq (q+2)(\mu_q - q).$$

REFERENCES

[1] J.A.Bondy and U.S.R.Murty, "Graph Theory with Applications". Macmillan, London and Elsevier, New York, 1976.

[2] Zh.G. Nikoghosyan, "Advanced Lower Bounds for the Circumference", Graphs and Combinatorics 29, pp. 1531-1541, 2013.