The Weighted Partial Vertex Cover Problem in Bipartite Graphs

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ABSTRACT

In the weighted partial vertex cover problem (WPVC), we are given a graph $G = (V, E)$, cost function $c : V \to N$, profit function $p : E \to N$, and positive integers R and L. The goal is to check whether there is a subset V' of cost at most R, such that the total profit of edges covered by V' is at least L. In this paper we study the fixed-parameter tractability (**FPT**) of WPVC in bipartite graphs (WPVCB). We prove that the question about fixed-parameter tractability of WPVCB problem is equivalent to its restriction when the input graph is a complete bipartite graph. We consider one special case of WPVCB, where we allow to take only one fractional vertex and prove that this problem is **FPT** with respect to R. We also prove that WPVCB is **FPT** with respect to L. Finally, for bounded-degree graphs we show that WPVC is **FPT** with respect to R.

Keywords

Vertex cover, Partial vertex cover, Bipartite graph, Fixed parameter tractability.

1. INTRODUCTION

In this paper, we consider finite, undirected graphs that have no loops or multiple edges. As usual, the degree of a vertex is the number of edges of the graph that are incident to it. The maximum degree of the graph G is just the maximum of all degrees of vertices of G.

A graph $G = (V, E)$ is bipartite, if its vertex set can be partitioned into 2 sets V_1 and V_2, so that an edge of G joins a vertex from V_1 to one from V_2. A bipartite graph G is said to be complete if any vertex from V_1 is joined to a vertex from V_2. If G is a complete bipartite graph with $|V_1| = m$ and $|V_2| = n$, then we will denote G as $K_{m,n}$.

Definition 1. *Given a graph $G = (V, E)$, and a set $S \subset V$ of vertices, an edge $(i, j) \in E$ is covered by S, if $i \in S$, or $j \in S$.*

Define $E(S)$ to be the set of edges of G that are covered with at least one vertex of S.

The classical Vertex Cover problem (VC) is defined as finding the smallest set S of vertices of the input graph G, so that $E(S) = E$. The vertex cover problem is a well-known **NP-complete** problem [21]. The Partial Vertex Cover problem represents a natural theoretical generalization of VC, which is also motivated by practical applications. Flow-based risk-assessment models in computational systems, for example, can be viewed as instances of PVC [8].

The Parital Vertex Cover problem on bipartite graphs is **NP-hard**. The computational complexity of this problem has been open and recently and independently shown to be **NP-hard** [3, 19, 9, 10]. Many 2-approximation algorithms for VC are known [29]. There is an approximation algorithm for the VC problem which has an approximation factor of $2 - \theta(\frac{1}{\sqrt{logn}})$ [20]. This is the best algorithm is date. The VC problem is also known to be **APX-complete** [25]. Moreover, it cannot be approximated within a factor of 1.3606 unless $\mathbf{P} = \mathbf{NP}$ [11] and not within any constant factor smaller than 2, unless the unique games conjecture is false [22]. Let us note that in [18], a $(\frac{4}{3} + \epsilon)$-approximation algorithm is designed for WPVCB for each $\epsilon > 0$.

All the hardness results for the VC problem directly apply to the PVC problem because the PVC problem is an extension of the VC problem. Since 1990s the PVC problem and the partial-cover variants of similar graph problems have been extensively studied [7], [6], [16], [15], [17], [23]. In particular, there is an $O(n \cdot \log n + m)$-time 2-approximation algorithm based on the primal-dual method [23], as well as another combinatorial 2-approximation algorithm [27]. Both of these algorithms are for a more general soft-capacitated version of PVC. There are several older 2-approximations resulting from different approaches [5], [7], [26], [13]. Let us also note that the WPVC problem for trees (WPVCT) is studied in [24], where the authors provide an **FPTAS** for it, and a polynomial time algorithm for the case when vertices have no weights.

Another problem with close relationship to WPVC is the Budgeted Maximum Coverage problem (BMC). In this problem one tries to find a min-cost (the profit of covered edges) is maximized. In some sense, this problem can be viewed as a problem "dual" to WPVC, and it can be shown that both problems are equivalent from the perspective of exact solvability. The BMC problem for sets (not necessarily graphs) admits a $(1 - \frac{1}{e})$-approximation algorithm [28] but, special cases that beat this bound are rare. The pipage rounding technique gives a $\frac{3}{4}$-approximation algorithm for the BMC problem on graphs [2] which is improved to $\frac{4}{5}$ for bipartite graphs [4]. Finally, let us note that in [9, 10], an 8/9-approximation algorithm for the problem is presented when the input graph is bipartite and the vertices are unweighted. The result is based on the linear-programming formulation of the problem, and the constant 8/9 matches the integrality gap of the linear program.

Recall that a combinatorial problem Π is said to be fixed-parameter tractable (**FPT**) with respect to a parameter k, if there is an algorithm for solving Π exactly, is bounded by $f(k) \cdot size^{O(1)}$. Here f is some (computable) function of k, and $size$ is the length of the input. From the perspective of fixed-parameter tractability the PVC problem is in some sense more difficult than the VC problem. For instance, the PVC problem is **W[1]-complete**, while the VC problem is fixed parameter tractable [14]. Related with this topic, let us note that in [1] it is shown that WPVCB is **FPT** with respect to R, when the vertices and edges of the graph are unweighted. By extending the methods of the authors, it is easy to show that the case when edges may have weights is also **FPT** with respect to R.

Below the formal definition of the main problem is given.

Definition 2. *Given an integer L, undirected bipartite graph $G = (V, E)$, and weight-functions $w : V \to N$ and $p : E \to N$. The WPVCB problem is defined as finding a subset $S \subset V$ such that $\sum_{e \in E(S)} p(e) \geq L$, and $\sum_{v \in S} w(v)$ is minimized among all such subsets of V.*

The main goal of the present paper is to investigate this problem from the perspective of fixed-parameter tractability.

2. MAIN RESULTS

In this section we present our main results. The first of them deals with the WPVCB's extension when the vertex and edge weights can be rational numbers. We show that this assumption does not increase the computational complexity of the problem.

Proposition 1. *WPVCB is equivalent to its extension when vertex and edge weights can be rational numbers.*

Our next result justifies why while considering WPVCB, one can assume that the input graph is a complete bipartite.

Proposition 2. *WPVCB is equivalent to its restriction when the input graph G is a complete bipartite graph $K_{m,n}$.*

Next, we consider a version of the partial vertex cover problem where some vertices may be taken fractionally. When a vertex v is taken to an extent of α, $(0 < \alpha < 1)$, we will assume that it contributes to the weight or cost of the cover by $\alpha \cdot w(v)$, and to the coverage by $\alpha \cdot p(v)$. If a vertex v is taken fractionally, we will say that v is a fractional vertex.

Lemma 1. *WPVCB is fixed-parameter tractable with respect to R when we consider a version of the WPVCB problem, where we allow to take only one fractional vertex.*

When the input graph is a union of stars.

Theorem 1. *WPVCB is fixed-parameter tractable with respect to R when the input graph is a union of stars.*

A class of graphs is said to be bounded-degree, if there is a constant C, such that all graphs from the class have maximum degrees at most C. It turns out that when the input graphs have bounded-degree and need not be bipartite, we have the following result:

Theorem 2. *WPVC is fixed-parameter tractable with respect to R when the input graph is bounded-degree.*

Finally, if we consider the same WPVCB problem with respect to another parameter, more precisely, L, then the following result holds:

Theorem 3. *WPVCB is fixed-parameter tractable with respect to L.*

REFERENCES

[1] Omid Amini, Fedor V. Fomin and Saket Saurabh, "Implicit branching and parameterized partial cover problems", *J. Comp. Sys. Sciences (77)*, pp. 1159–1171, 2011.

[2] Alexander A. Ageev, Maxim Sviridenko, "Approximation algorithms for maximum coverage and max cut with given size of parts", *IPCO*, pp. 17–30, 1999.

[3] N. Apollonio, B. Simeone, "The maximum vertex coverage problem on bipartite graphs", *Discrete Appl. Math. 165*, pp. 37–48, 2014.

[4] N. Apollonio, B. Simeone, "Improved approximation of maximum vertex coverage problem on bipartite graphs", *SIAM J. Discrete Math. 28(3)*, pp. 1137–1151, 2014.

[5] Reuven Bar-Yehuda, "Using homogeneous weights for approximating the partial cover problem", *J. Algorithms 39(2)*, pp. 137–144, 2001.

[6] Markus Bläser, "Computing small partial coverings", *Inf. Process. Lett 85(6)*, pp. 327–331, 2003.

[7] Nader H. Bshouty, Lynn Burroughs, "Massaging a linear programming solution to give a 2-approximation for a generalization of the vertex cover problem", In *Proceedings of* STACS, pp. 298–308, 1998.

[8] C. Cagatay Bilgin B. Caskurlu, A. Gehani, K. Subramani, "Analytical models for risk-based intrusion response", *Computer Networks(Special issue on Security/Identity Architecture) 57(10)*, pp. 2181–2192, 2013.

[9] B. Caskurlu, V. Mkrtchyan, O. Parekh, K. Subramani, "Partial Vertex Cover and Budgeted Maximum Coverage in Bipartite Graphs", *SIAM J. Disc. Math.*, 2017, to appear.

[10] B. Caskurlu, V. Mkrtchyan, O. Parekh, K. Subramani, "On Partial Vertex Cover and Budgeted Maximum Coverage Problems in Bipartite Graphs", *IFIP TCS*, pp. 13–26, 2014.

[11] Irit Dinur, Samuel Safra, "On the hardness of approximating minimum vertex cover", *Ann. of Math. 162(1)*, pp. 439–485, 2005.

[12] Uriel Feige, "A threshhold of $ln(n)$ for approximating set cover", In *Proceedings of the 28th Annual ACM Symposium on the theory of Computing*, pp. 314–318, 1996.

[13] Dorit S. Hochbaum, "The t-vertex cover problem: Extending the half integrality framework with budget constraints", In *Proceedings of APPROX*, pp. 111–122, 1998.

[14] Rolf Niedermeier Jiong Guo, Sebastian Wernicke, "Parameterized complexity of generalized vertex cover problems", *Lecture Notes in Computer Science 3608*, pp. 36–48, 2005.

[15] Alexander Langer Joachim Kneis, Peter Rossmanith, "Improved upper bounds for partial vertex cover", In *In WG*, pp. 240–251, 2008.

[16] Daniel Mölle Joachim Kneis, Peter Rossmanith, "Partial vs. complete domination: t-dominating set", In *In SOFSEM (1)*, pp. 367–376, 2007.

[17] Stefan Richter Joachim Kneis, Daniel Mölle, Peter Rossmanith, "Intuitive algorithms and t-vertex cover", In *In ISAAC*, pp. 598–607, 2006.

[18] Ojas Parekh Jochen Könemann, Danny Segev, "A unified approach to approximating partial covering problems", *Algorithmica 59(4)*, pp. 489–509, 2011.

[19] G. Joret, A. Vetta, "Reducing the Rank of a Matroid", *arXiv:1211.4853v1*, [cs.DS], 2012.

[20] George Karakostas, "A better approximation ratio for the vertex cover problem", *ACM Transactions on Algorithms 5(4)*, 2009.

[21] R. Karp, "Reducibility among combinatorial problems", In *In R. Miller and J. Thatcher, editors, Complexity of Computer Computations*, pp. 85–103, Plenum Press, 1972.

[22] Subhash Khot, Oded Regev, "Vertex cover might be hard to approximate to within $2-\epsilon$, *J. Comput. Syst. Sci. 74*, pp. 335–349, May 2008.

[23] Julian Mestre, "A primal-dual approximation algorithm for partial vertex cover: Making educated guesses", *Algorithmica 55(1)*, pp. 227–239, 2009.

[24] V. Mkrtchyan, O. Parekh, D. Segev, K. Subramani, "The approximability of Partial vertex covers in trees", *Proceedings of the* 43^{rd} *International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM), Limerick, Ireland*, pp. 350-360, 2017.

[25] Christos H. Papadimitriou, Mihalis Yannakakis, "Optimization, approximation, and complexity classes", *J. Comput. System Sci. 43(3)*, pp. 425–440, 1991.

[26] Samir Khuller Rajiv Gandhi, Aravind Srinivasan, "Approximation algorithms for partial covering problems", *J. Algorithms 53(1)*, pp. 55–84, 2004.

[27] Julián Mestre Reuven Bar-Yehuda, Guy Flysher, Dror Rawitz, "Approximation of partial capacitated vertex cover", *Lecture Notes in Computer Science 4698*, pp. 335–346, 2007.

[28] Anna Moss Samir Khuler, Joseph (Seffi) Naor, "The budgeted maximum coverage problem", *Inform. Process. Lett. 70(1)*, pp. 39–45, 1999.

[29] Vijay V. Vazirani, "*Approximation Algorithms*", Springer-Verlag New York, Inc., New York, NY, USA, 2001.