On the Palette Index of Bipartite Graphs

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ABSTRACT

A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \rightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the *palette of a vertex* v, denoted by $P(v, \alpha)$, is the set of all colors appearing on edges incident to v. The *palette index* of a graph G, denoted by $\check{s}(G)$, is the minimum number of distinct palettes taken over all proper edge-colorings of G. In this paper we investigate the palette index of bipartite graphs. In particular, we prove that: 1) if G is a bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \leq 11$, and moreover if G is a bipartite graph with $\Delta(G) = 4$ and without pendant vertices, then $\check{s}(G) \leq 7$; 2) if G is an Eulerian bipartite graph with $\Delta(G) \leq 6$, then $\check{s}(G) \leq 7$; 3) if G is an Eulerian bipartite graph with $\Delta(G) = 8$, then $\check{s}(G) \leq 13$. We also obtain some results on the palette index of (a, b)-biregular bipartite graphs. In particular, we prove that if G is a $(2, 2r)$-biregular ($r \geq 2$) bipartite graph, then $\check{s}(G) = r + 1$, if G is a $(2, 2r + 1)$-biregular ($r \in \mathbb{N}$) bipartite graph, then $r+2 \leq \check{s}(G) \leq 2r+2$, and if G is a $(2r-2, 2r)$-biregular ($r \geq 2$) bipartite graph, then $\check{s}(G) = r + 1$.

Keywords

Edge-coloring, palette index, bipartite graph, biregular bipartite graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The maximum degree of vertices in G is denoted by $\Delta(G)$ and the chromatic index of G by $\chi'(G)$. A graph G is Eulerian if it has a closed trail containing every edge of G. An (a, b)-biregular bipartite graph G is a bipartite graph G with the vertices in one part all having degree a and the vertices in the other part all having degree b. The terms and concepts that we do not define can be found in [1, 6, 10].

A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \rightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If α is a proper edge-coloring of a graph G and $v \in V(G)$, then the *palette of a vertex* v, denoted by $P(v, \alpha)$, is the set of all colors appearing on edges incident to v. For a proper edge-coloring α of a graph G, we define $P(G, \alpha)$ as follows: $P(G, \alpha) = \{P(v, \alpha) : v \in V(G)\}$. Clearly, for every graph G and its proper edge-coloring α, we have $1 \leq |P(G, \alpha)| \leq |V(G)|$. In [4], Burris and Schelp introduced the concept of vertex-distinguishing proper edge-colorings of graphs. A proper edge-coloring α of a graph G is a vertex-distinguishing proper edge-coloring if for every pair of distinct vertices u and v of G, $P(u, \alpha) \neq P(v, \alpha)$. This means that if α is a vertex-distinguishing proper edge-coloring of G, then $|P(G, \alpha)| = |V(G)|$. In some papers [2, 9, 11], the authors considered the special case of vertex-distinguishing proper edge-colorings of graphs which is called an adjacent vertex-distinguishing edge-coloring. A proper edge-coloring α of a graph G is adjacent vertex-distinguishing if for every pair of adjacent vertices u and v of G, $P(u, \alpha) \neq P(v, \alpha)$. On the other hand, Horňák, Kalinowski, Meszka and Woźniak [8] initiated the investigation of the problem of finding proper edge-colorings of graphs with the minimum number of distinct palettes. For a graph G, we define the *palette index* $\check{s}(G)$ of a graph G as follows: $\check{s}(G) = \min_{\alpha} |P(G, \alpha)|$, where minimum is taken over all possible proper edge-colorings of G. In [8], the authors proved that $\check{s}(G) = 1$ if and only if G is regular and $\chi'(G) = \Delta(G)$. From here and the result of Holyer [7] it follows that for a given regular graph G, the problem of determining whether $\check{s}(G) = 1$ or not is NP-complete. Moreover, they also proved that if G is regular, then $\check{s}(G) \neq 2$. In [8], Horňák, Kalinowski, Meszka and Woźniak determined the palette index of complete and cubic graphs. In particular, they showed that

$$\check{s}(K_n) = \begin{cases} 1, \text{ if } n \equiv 0 \pmod 2 \text{ or } n = 1, \\ 3, \text{ if } n \equiv 3 \pmod 4, \\ 4, \text{ if } n \equiv 1 \pmod 4. \end{cases}$$

Recently, Bonvicini and Mazzuoccolo [3] studied the palette index of 4-regular graphs. In particular, they constructed 4-regular graphs with palette index 4 and 5.

In this paper, we study the palette index of bipartite graphs. In particular, we obtain some results on the palette index of bipartite graphs with small maximum degree. We also obtain some results on the palette index of (a, b)-biregular bipartite graphs. In particular, we prove that if G is a $(2, 2r)$-biregular ($r \geq 2$) bipartite graph, then $\check{s}(G) = r + 1$, if G is a $(2, 2r + 1)$-biregular

$(r \in \mathbb{N})$ bipartite graph, then $r+2 \le \check{s}(G) \le 2r+2$, and if G is a $(2r-2, 2r)$-biregular $(r \ge 2)$ bipartite graph, then $\check{s}(G) = r+1$.

2. PALETTE INDEX OF BIPARTITE GRAPHS WITH SMALL MAXIMUM DEGREE

The well-known Kőnig's edge coloring theorem states that $\chi'(G) = \Delta(G)$ for any bipartite graph G. This implies that if G is a bipartite graph, then $\check{s}(G) \le 2^{\Delta(G)} - 1$. Moreover, for a bipartite graph G, $\check{s}(G) = 1$ if and only if G is regular. So, the palette index of a non-regular bipartite graph G satisfies the following inequalities: $2 \le \check{s}(G) \le 2^{\Delta(G)} - 1$. In particular, if G is a bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \le 15$. Here we improve this upper bound.

Theorem 1. *If G is a bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \le 11$. Moreover, if G is a bipartite graph with $\Delta(G) = 4$ and without pendant vertices, then $\check{s}(G) \le 7$.*

Theorem 2. *If G is an Eulerian bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \le 3$.*

Next we consider bipartite graphs with maximum degree 5. Clearly, for every bipartite graph G with $\Delta(G) = 5$, $\check{s}(G) \le 31$.

Theorem 3. *If G is a bipartite graph with $\Delta(G) = 5$ that has a perfect matching, then $\check{s}(G) \le 12$.*

Theorem 4. *If G is a bipartite graph with $\Delta(G) = 5$, then $\check{s}(G) \le 23$. Moreover, if G is a bipartite graph with $\Delta(G) = 5$ and without a vertex of degree 3, then $\check{s}(G) \le 17$.*

We also consider Eulerian bipartite graphs with maximum degree 6 and 8, respectively.

Theorem 5. *If G is an Eulerian bipartite graph with $\Delta(G) = 6$, then $\check{s}(G) \le 7$.*

Theorem 6. *If G is an Eulerian bipartite graph with $\Delta(G) = 8$, then $\check{s}(G) \le 13$.*

3. PALETTE INDEX OF BIREGULAR BIPARTITE GRAPHS

In this section we consider the palette index of (a,b)-biregular bipartite graphs. Since for an a-regular bipartite graph G, $\chi'(G) = \Delta(G) = a$, we obtain that $\check{s}(G) = 1$. So, without loss of generality we may assume that $a < b$. Kőnig's edge coloring theorem implies that $\check{s}(G) \le 1 + \binom{b}{a}$ for any (a,b)-biregular bipartite graph G. In particular, for any $(b-1, b)$-biregular bipartite graph G, we obtain $\check{s}(G) \le 1 + b$. On the other hand, it is not difficult to see that $\check{s}(G) \ge 1 + \lceil \frac{b}{a} \rceil$.

First we consider $(2,b)$-biregular bipartite graphs.

Theorem 7. *If G is a $(2, 2r)$-biregular $(r \ge 2)$ bipartite graph, then $\check{s}(G) = r+1$.*

Theorem 8. *If G is a $(2, 2r+1)$-biregular $(r \in \mathbb{N})$ bipartite graph, then $r+2 \le \check{s}(G) \le 2r+2$.*

Let us note that the upper bound in Theorem 8 is sharp, since for any $(2,3)$-biregular bipartite graph G, $\check{s}(G) = 4$.

Next we consider $(2r-2, 2r)$-biregular bipartite graphs.

Theorem 9. *If G is a $(2r-2, 2r)$-biregular $(r \ge 2)$ bipartite graph, then $\check{s}(G) = r+1$.*

We also consider some special cases of (a,b)-biregular bipartite graphs.

Theorem 10. *If G is a $(3,5)$-biregular bipartite graph, then $3 \le \check{s}(G) \le 7$.*

Theorem 11. *If G is a $(3,6)$-biregular bipartite graph, then $3 \le \check{s}(G) \le 5$.*

Theorem 12. *If G is a $(3,9)$-biregular bipartite graph, then $4 \le \check{s}(G) \le 10$.*

Theorem 13. *If G is a $(4,8)$-biregular bipartite graph, then $3 \le \check{s}(G) \le 5$.*

Theorem 14. *If G is a $(5,10)$-biregular bipartite graph, then $3 \le \check{s}(G) \le 9$.*

One of the special cases of (a,b)-biregular bipartite graphs is the complete bipartite graph $K_{a,b}$. For complete bipartite graphs, we prove that the following result holds.

Theorem 15. *If $a < b$ $(a, b \in \mathbb{N})$, then*

$$1 + \left\lceil \frac{b}{a} \right\rceil \le \check{s}\left(K_{a,b}\right) \le 1 + \frac{b}{\gcd(a,b)}.$$

Corollary 16. *If $\gcd(a,b) = a$ $(a < b)$, then $\check{s}\left(K_{a,b}\right) = 1 + \frac{b}{a}$.*

Finally, we would like to point out the connection between cyclic interval colorings of (a,b)-biregular bipartite graphs and the palette index of these graphs. Recall that a proper t-edge-coloring $\alpha : E(G) \to \{1, \dots, t\}$ of a graph G is called a *cyclic interval t-coloring* if the colors of edges incident to every vertex v of G either form an interval of integers or the set $\{1, \dots, t\} \setminus \{\alpha(e) : e \text{ is incident to } v\}$ is an interval of integers.

In [5], Casselgren and Toft suggested the following conjecture on (a,b)-biregular bipartite graphs.

Conjecture 17. *Every (a, b)-biregular bipartite graph has a cyclic interval $\max\{a, b\}$-coloring.*

If the conjecture is true, then it is not difficult to see that $\check{s}(G) \leq 1 + \max\{a, b\}$ for every (a, b)-biregular bipartite graph G. So, we would like to suggest the following conjecture.

Conjecture 18. *For any (a, b)-biregular bipartite graph G, $\check{s}(G) \leq 1 + \max\{a, b\}$.*

Kőnig's edge coloring theorem implies that this conjecture is true for all $(b-1, b)$-biregular bipartite graphs. As we show above the conjecture is also true for all $(2, b)$-biregular, $(2r-2, 2r)$-biregular ($r \geq 2$) and (a, b)-biregular ($(a, b) \in \{(3, 6), (3, 9), (4, 8), (5, 10)\}$) bipartite graphs.

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