On Locally-Balanced 2-Partitions of Some Graphs

Aram H. Gharibyan
Department of Informatics and Applied Mathematics,
Yerevan State University, 0025, Yerevan, Armenia
e-mail: aramgharibyan@gmail.com

Petros A. Petrosyan
Institute for Informatics and Automation Problems,
National Academy of Sciences of Armenia,
0014, Yerevan, Armenia
Department of Informatics and Applied Mathematics,
Yerevan State University, 0025, Yerevan, Armenia
e-mail: pet_petros@ipia.sci.am,
petros_petrosyan@ysu.am

ABSTRACT

A 2-partition of a graph G is a function $f : V(G) \to \{0,1\}$. A 2-partition f of a graph G is locally-balanced with an open neighborhood if for every $v \in V(G)$, $||\{u \in N_G(v) : f(u) = 1\}| - |\{u \in N_G(v) : f(u) = 0\}|| \leq 1$, where $N_G(v) = \{u \in V(G)\colon uv \in E(G)\}$. A 2-partition f' of a graph G is locally-balanced with a closed neighborhood if for every $v \in V(G)$, $||\{u \in N_G[v] : f'(u) = 1\}| - |\{u \in N_G[v] : f'(u) = 0\}|| \leq 1$, where $N_G[v] = N_G(v) \cup \{v\}$. In this paper we obtain some conditions for the existence of locally-balanced 2-partitions of certain graphs. In particular, we prove some necessary condition for the existence of locally-balanced 2-partitions of Eulerian graphs. Moreover, we also obtain some results on the existence of locally-balanced 2-partitions of rook's graphs and powers of cycles.

Keywords

Locally-balanced 2-partition, equitable coloring, Eulerian graph, rook's graph, power of cycles.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. Let $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the maximum degree of vertices in G by $\Delta(G)$. A graph G is odd if the degree of every vertex of G is odd. A graph G is Eulerian if it has a closed trail containing every edge of G. We use the standard notations C_n and K_n for the simple cycle and the complete graph on n vertices, respectively. A graph is a power of cycle, denoted C_n^k, if $V\left(C_n^k\right) = \{v_0, \ldots, v_{n-1}\}$ and $E\left(C_n^k\right) = E_1 \cup \cdots \cup E_k$, where $E_i = \{v_j v_{(j+i) \pmod n} : 0 \leq j \leq n-1\}$. Clearly, C_n^k is a $2k$-regular ($k \in \mathbb{N}$) graph. The terms and concepts that we do not define can be found in [6, 13].

Let G and H be graphs. The Cartesian product $G \square H$ of graphs G and H is defined as follows:

$$V(G \square H) = V(G) \times V(H),$$
$$E(G \square H) = \{(u_1, v_1)(u_2, v_2)\colon (u_1 = u_2 \wedge v_1 v_2 \in E(H)) \vee (v_1 = v_2 \wedge u_1 u_2 \in E(G))\}.$$

The Cartesian product $K_m \square K_n$ of two complete graphs K_m and K_n is called a rook's graph.

A 2-partition of a graph G is a function $f : V(G) \to \{0,1\}$. A 2-partition f of a graph G is locally-balanced with an open neighborhood if for every $v \in V(G)$, $||\{u \in N_G(v)\colon f(u) = 1\}| - |\{u \in N_G(v)\colon f(u) = 0\}|| \leq 1$. A 2-partition f' of a graph G is locally-balanced with a closed neighborhood if for every $v \in V(G)$, $||\{u \in N_G[v]\colon f'(u) = 1\}| - |\{u \in N_G[v]\colon f'(u) = 0\}|| \leq 1$. The concept of locally-balanced 2-partition of graphs was introduced by Balikyan and Kamalian [10] in 2005, and it can be considered as a special case of equitable colorings of hypergraphs [1]. In [1], Berge obtained some sufficient conditions for the existence of equitable colorings of hypergraphs. In [5, 7, 8, 12], the authors considered the problems of the existence and construction of proper vertex-coloring of a graph for which the number of vertices in any two color classes differ by at most one. In [9], 2-vertex-colorings of graphs were considered for which each vertex is adjacent to the same number of vertices of every color. In particular, in [9], it was proved that the problem of the existence of such a coloring is NP-complete even for the $(2p, 2q)$-biregular $(p, q \geq 2)$ bipartite graphs. In [10], Balikyan and Kamalian proved that the problem of existence of locally-balanced 2-partition with an open neighborhood of bipartite graphs with maximum degree 3 is NP-complete. Later, they also proved [11] the similar result for locally-balanced 2-partitions with a closed neighborhood. In [2, 3], the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of trees were obtained. In [4], the authors obtained the necessary and sufficient conditions for the existence of locally-balanced 2-partitions of complete multipartite graphs.

In the present paper we obtain some conditions for the existence of locally-balanced 2-partitions of certain graphs. In particular, we prove some necessary condition for the existence of locally-balanced 2-partitions of Eulerian graphs. Moreover, we also obtain some results on the existence of locally-balanced 2-partitions of rook's graphs and cycle powers.

2. THE MAIN RESULTS

We first prove the following two results.

Theorem 1. *Let G be an Eulerian graph and $k = \min\{q : v \in V(G), d_G(v) = p2^q, where\ p\ is\ odd\ and\ q \in \mathbb{Z}_{\geq 0}\}$. If G has a locally-balanced 2-partition with an open neighborhood, then*

$|\{v : v \in V(G), d_G(v) = p2^k, where\ p\ is\ odd\}|$ *is even.*

Corollary 2. *Every $2k$-regular graph of odd order has no locally-balanced 2-partition with an open neighborhood.*

Theorem 3. *Let G be an odd graph and $k = \min\{q : v \in V(G), d_G(v) + 1 = p2^q, where\ p\ is\ odd\ and\ q \in \mathbb{Z}_{\geq 0}\}$. If G has a locally-balanced 2-partition with a closed neighborhood, then*

$|\{v : v \in V(G), d_G(v) + 1 = p2^k, where\ p\ is\ odd\}|$ *is even.*

Next we consider rook's graphs. For these graphs we prove the following results.

Theorem 4. *If $m, n \geq 2$, then the graph $K_m \square K_n$ has a locally-balanced 2-partition with a closed neighborhood if and only if m and n are even.*

Theorem 5. *If either m and n are odd and $m > 2$ or m and n are even and $m, n > 2$, then the graph $K_m \square K_n$ has no locally-balanced 2-partition with an open neighborhood.*

Finally, we consider powers of cycles. For these graphs we prove the following results.

Theorem 6. *If n is odd ($n, k \in \mathbb{N}$), then C_n^k has no locally-balanced 2-partition with an open neighborhood.*

Theorem 7. *If n and k are even ($n, k \in \mathbb{N}$), then C_n^k has a locally-balanced 2-partition with an open neighborhood.*

Theorem 8. *If n and $\frac{n}{k+1}$ are even ($n, k \in \mathbb{N}$), then C_n^k has a locally-balanced 2-partition with an open neighborhood.*

Theorem 9. *If n is even, k is odd and $\frac{lcm(n,k+1)}{k+1}$ is odd ($n, k \in \mathbb{N}$), then C_n^k has no locally-balanced 2-partition with an open neighborhood.*

REFERENCES

[1] C. Berge, Graphs and Hypergraphs, Elsevier Science Ltd, 1985.

[2] S.V. Balikyan, "On existence of certain locally-balanced 2-partition of a tree", *Mathematical Problems of Computer Science 30*, pp. 25-30, 2008.

[3] S.V. Balikyan, R.R. Kamalian, "On existence of 2-partition of a tree, which obeys the given priority", *Mathematical Problems of Computer Science 30*, pp. 31-35, 2008.

[4] A.H. Gharibyan, P.A. Petrosyan, "Locally-balanced 2-partitions of complete multipartite graphs", *6th Polish Combinatorial Conference, Bedlewo, Poland*, p. 19, 2016.

[5] A. Hajnal, E. Szemeredi, "Proof of a conjecture of P. Erdős", *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfred, 1969), North-Holland*, pp. 601-623, 1970.

[6] G. Chartrand, P. Zhang, Chromatic Graph Theory, Discrete Mathematics and Its Applications, CRC Press, 2009.

[7] W. Meyer, "Equitable coloring", *American Mathematical Monthly 80 (8)*, pp. 920-922, 1973.

[8] A.V. Kostochka, "Equitable colorings of outerplanar graphs", *Discrete Mathematics 258*, pp. 373-377, 2002.

[9] J. Kratochvil, "Complexity of hypergraph coloring and Seidel's switching", *Graph Theoretic Concepts in Computer Science, 29th International Workshop, WG 2003, Elspeet, The Netherlands, Revised Papers, volume 2880*, pp. 297-308, June 19-21, 2003.

[10] S.V. Balikyan, R.R. Kamalian, "On NP-completeness of the problem of existence of locally-balanced 2-partition for bipartite graphs G with $\Delta(G) = 3$", *Doklady NAN RA 105 (1)*, pp. 21-27, 2005.

[11] S.V. Balikyan, R.R. Kamalian, "On NP-completeness of the problem of existence of locally-balanced 2-partition for bipartite graphs G with $\Delta(G) = 4$ under the extended definition of the neighbourhood of a vertex", *Doklady NAN RA 106 (3)*, pp. 218-226, 2006.

[12] D. de Werra, "On good and equitable colorings", *In Cahiers du C.E.R.O., volume 17*, pp. 417-426, 1975.

[13] D.B. West, Introduction to Graph Theory, Prentice-Hall, New Jersey, 2001.