Maximum Inequalities and their Applications to Hadamard Matrices

George Giorgobiani
Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University
Tbilisi, Georgia
e-mail: giorgobiani.g@gtu.ge

Vakhtang Kvaratskhelia
Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University
Tbilisi, Georgia
e-mail: v.kvaratskhelia@gtu.ge

Marine Menteshashvili
Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University
Tbilisi, Georgia
e-mail: m.menteshashvili@gtu.ge

ABSTRACT

A new numerical characterization for Hadamard matrices is introduced. Its estimations for different norms are established by use of appropriate maximal inequalities for the signed vector summands.

Keywords

Maximal inequalities, Hadamard matrices, Chernoff bound.

1. INTRODUCTION

In what follows $\mathbb{R}^n$, $1 \le n < \infty$, denotes the $n-dimensional$ normed vector space over the field of real numbers. $\mathbb{R}^n$ will be equipped with the l_p-norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

or with the maximum norm (the case $p = \infty$)

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\},$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.

Let $H_n = [h_{ij}^n]$, $i, j = 1,2,\dots,n$; $1 < n < \infty$, be a Hadamard matrix of order n, i.e., the rows (and, hence, the columns) are mutually orthogonal and the entries all equal to ± 1. Thus,

$$h_{ij}^n = \pm 1 \quad and \quad H_n H_n^T = n\, I_n,$$

where the superscript T means transposition and I_n is the identity matrix of order n.

According to the work [7], let us denote by $\mathbb{N}_{\mathcal{H}}$ the subset of all positive integers $\mathbb{N}$ for which there exists a Hadamard matrix of order n. Let $\mathcal{H}_n^{all}$ be the set of all Hadamard matrices of order $n, n \in \mathbb{N}_{\mathcal{H}}$. Let further $1 \le p \le \infty$. For a Hadamard matrix $H_n = [h_{ij}^n]$, $i, j = 1,2,\dots,n; 1 < n < \infty$, consider the following functionals:

$$\varrho_{p,H_n} = \max_{1\le m\le n} \left(\sum\nolimits_{j=1}^{n} \left|\sum\nolimits_{i=1}^{m} h_{ij}^n\right|^p\right)^{1/p}, \quad m = 1,2,\cdots,n,$$

$$\varrho_{p,n} = \max_{H_n \in \mathcal{H}_n^{all}} \varrho_{p,H_n}.$$

In the same paper (see also [3,4,5]) the following estimates have been obtained:

$$\frac{1}{\sqrt{2}} \cdot n^{(p+2)/2p} \le \varrho_{p,n} \le n^{(p+2)/2p}, \quad for \quad 1 \le p \le 2, \ (1)$$

and

$$\varrho_{p,n} = n, \qquad for\ 2 \le p \le \infty. \ (2)$$

Let us put the opposite question. How small the quantity ϱ_{p,H_n} can be? In other words we pose the problem to estimate the quantity

$$\gamma_{p,n} = \min_{H_n \in \mathcal{H}_n^{all}} \varrho_{p,H_n}.$$

Clearly, one way to treat this question is to alter the signs of the rows of the matrix H_n, or one can consider the rearrangements of rows as well. The class of Hadamard matrices is closed under these transformations.

Denote the rows of H_n by $h_i, i = 1,\dots,n$. Then it's easy to see that

$$\varrho_{p,H_n} = \max_{1\le m\le n} \left\|\sum\nolimits_{i=1}^{m} h_i\right\|_p.$$

The main goal of this work is to establish the above bounds for the minimum with respect to the collections of signs $\vartheta = (\vartheta_1, \cdots, \vartheta_n)$, $\vartheta_i = \pm 1$, for the functional

$$\varrho_{p,H_n}(\vartheta) = \max_{1\le m\le n} \left\|\sum\nolimits_{i=1}^{m} \vartheta_i h_i\right\|_p, \ 1 \le p \le \infty,$$

where H_n is the given Hadamard matrix.

Let us first consider some maximum inequalities for the collections of signs.

Below we make use of the Hoeffding's inequality for Rademacher sums (see e.g., Theorem 2 of [6]):

For the real numbers $x_1, x_2, \cdots, x_n$, the following is true

$$\lambda\left\{u \in [0,1]: |\textstyle\sum_{i=1}^n x_i r_i(u)| > t(\sum_{i=1}^n x_i^2)^{\frac{1}{2}}\right\} \le 2exp\left(-\frac{1}{2}t^2\right). (3)$$

Here $(r_i)_{i=1}^n$ is the Rademacher system

$$r_i(u) = sign \sin 2^i \pi t, \ \ u \in [0,1], i \in \mathbb{N},$$

and λ denotes the Lebesgue measure on $[0,1]$.

2. MAIN RESULTS

For the case $1 \le p < \infty$ we get the following estimate.

Proposition 1. *Let $a_1, a_2, \cdots, a_m$ be the elements of $\mathbb{R}^n, 1 \le p < \infty,\ 1 < n < \infty,\ a_i = (a_{i1}, a_{i2}, \cdots, a_{in})$. Then there exists a collection of signs $\vartheta \in \{-1,+1\}^m$ such that*

$$\max_{1\le k\le m} \left\|\textstyle\sum_{i=1}^k \vartheta_i a_i\right\|_p \le C_p \max_{1\le j\le n} \|b_j\|_s, \quad (4)$$

where $C_p = \sqrt[p]{n}\sqrt{7\ ln\ n}$, $b_j \equiv (a_{1j}, a_{2j}, \cdots, a_{mj})$, $j = 1,2,\dots,n$ *and* $s = min(p, 2)$

Proof. For $s = \min(p,2)$, choose $l_s, 1 \le l_s \le n$, dependent on s such that $\|b_{l_s}\|_s = \max_{1\le j\le m} \|b_j\|_s$. Clearly $\|b_j\|_2 \le \|b_{l_s}\|_s$ for each $j = 1,2,\dots,m$. Using Levy's inequality (see e.g., [9]) we have

$$\lambda\left\{\max_{1\le k\le m} \left\|\sum_{i=1}^{k} a_i r_i\right\|_p \le C_p \|b_{l_s}\|_s\right\}$$

$$\ge 1 - 2\lambda\left\{\left\|\sum_{i=1}^{m} a_i r_i\right\|_p > C_p \|b_{l_s}\|_s\right\}.$$

We have

$$\lambda\left\{\left\|\sum_{i=1}^{m} a_i r_i\right\|_p > C_p \|b_{l_s}\|_s\right\}$$

$$= \lambda\left\{\left(\sum_{j=1}^{n}\left|\sum_{i=1}^{m} a_{ij} r_i\right|^p\right)^{\frac{1}{p}} > C_p \|b_{l_s}\|_s\right\} =$$

$$= \lambda\left\{\sum_{j=1}^{n}\left|\sum_{i=1}^{m} a_{ij} r_i\right|^p > C_p{}^p \|b_{l_s}\|_s{}^p\right\} \leq$$

$$\leq \lambda\left[\bigcup_{j=1}^{n}\left(\left|\sum_{i=1}^{m} a_{ij} r_i\right|^p > \frac{C_p{}^p}{n}\|b_{l_s}\|_s{}^p\right)\right] =$$

$$= \lambda\left[\bigcup_{j=1}^{n}\left(\left|\sum_{i=1}^{m} a_{ij} r_i\right| > \frac{C_p}{\sqrt[p]{n}}\|b_{l_s}\|_s\right)\right]$$

$$\leq \lambda\left[\bigcup_{j=1}^{n}\left(\left|\sum_{i=1}^{m} a_{ij} r_i\right| > \frac{C_p}{\sqrt[p]{n}}\|b_j\|_2\right)\right].$$

Thus we get

$$\lambda\left\{\max_{1\leq k\leq m}\left\|\sum_{i=1}^{k} a_i r_i\right\|_p \leq C_p \|b_{l_s}\|_s\right\} \geq$$

$$1 - 2\sum_{j=1}^{n}\lambda\left(\left|\sum_{i=1}^{m} a_{ij} r_i\right| > \frac{C_p}{\sqrt[p]{n}}\|b_j\|_2\right) =$$

$$= 1 - 2\sum_{j=1}^{n}\lambda\left(\left|\sum_{i=1}^{m} a_{ij} r_i\right| > \sqrt{7\ \ln n}\|b_j\|_2\right).$$

Finally, using (3) we get for any $n \geq 2$

$$\lambda\left\{\max_{1\leq k\leq m}\left\|\sum_{i=1}^{k} a_i r_i\right\|_p \leq C_p \|b_{l_s}\|_s\right\}$$

$$\geq 1 - 4n\, exp\left(-\frac{7 ln\, n}{2}\right) = 1 - \frac{4}{n^{2.5}} > 0.$$

This implies the proof of the proposition.

For the case $p = \infty$ we refer to the well known result of Spencer:

Proposition 2 (J. Spencer, [8]**).** *Let $a_1, a_2, \cdots, a_n$ be the elements of $\mathbb{R}^n$. Then there exists a collection of signs $\vartheta \in \{-1, +1\}^n$ such that for some absolute constant K*

$$\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\vartheta_i a_i\right\|_\infty \leq K\sqrt{n}. \qquad (5)$$

Remark. Note that the Spencer's result is valid for exactly n vectors in n-dimensional space l_∞^n. For the maximum norm and any number m of vectors $a_1, a_2, \cdots, a_m$, the method of proof of Proposition 1 gives the following estimation

$$\max_{1\leq k\leq m}\left\|\sum_{i=1}^{k}\vartheta_i a_i\right\|_\infty \leq \sqrt{5\ ln\, n}\max_{1\leq j\leq d}\|b_j\|_2.$$

The similar inequality, related to the Discrepancy Theory, has been known earlier (see [1] Theorem 1.1, pg. 178).

Propositions 1 and 2 imply the following upper bounds for $\varrho_{p,H_n}(\vartheta)$.

Proposition 3. *Let $H_n = [h_{ij}^n]_{i,j=1}^n$ be the Hadamard matrix of order n. Then there exists a collection of signs $\vartheta_i = \pm 1, i = 1, \ldots, n$ such that for the Hadamard matrix $H_n^\vartheta = [\vartheta_i h_{ij}^n]_{i,j=1}^n$ the following estimates hold:*

a) when $1 < p \leq 2$

$$\varrho_{p,H_n}(\vartheta) \leq \sqrt{7}\ n^{\frac{2}{p}}\sqrt{ln\, n}\ ;$$

b) when $2 < p < \infty$

$$\varrho_{p,H_n}(\vartheta) \leq \sqrt{7}\ n^{\left(\frac{1}{p}+\frac{1}{2}\right)}\sqrt{ln\, n}\ ;$$

c) when $p = \infty$ for an absolute constant K

$$\varrho_{p,H_n}(\vartheta) \leq K\sqrt{n}\ .$$

As we see this method does not give any meaningful result for the case $1 < p \leq 2$. But the bounds in b) and especially in c) show that $\varrho_{p,H_n}(\vartheta)$ can be reduced by the selection of signs. It would be interesting to find the exact value of the constant K in c).

Proposition 3 immediately implies the following estimations for $\gamma_{p,n}$.

Corollary. *The following estimations are valid:*

a) when $2 < p < \infty$

$$\gamma_{p,n} \leq \sqrt{7}\ n^{\left(\frac{1}{p}+\frac{1}{2}\right)}\sqrt{ln\, n}\ ;$$

b) when $p = \infty$, for an absolute constant K

$$\gamma_{p,n} \leq K\sqrt{n}\ .$$

Let us note finally that, as it was mentioned above, one can investigate the behavior of ϱ_{p,H_n} for the permutations of rows as well. In this case a powerful transference technique, developed by S. Chobanyan (see e.g. [2]), could be applied. This may be the topic for the further research.

REFERENCES

[1] N. Alon, J. Spencer. "The Probabilistic Method." *J. Wiley and Sons, New York, NY,* 1992.

[2] S. A. Chobanyan, G. J. Georgobiani. "A problem on rearrangements of summands in normed spaces and Rademacher sums". *Probability theory on vector spaces, IV (Lancut, 1987), Lecture Notes in Math., Springer, Berlin,* 1391, pp. 33-46, 1989.

[3] A. Figula, V. Kvaratskhelia. "Some numerical characteristics of Sylvester and Hadamard matrices". *Publ. Math. Debrecen,* 86/1-2, pp. 149–168, 2015.

[4] G. Giorgobiani, V. Kvaratskhelia, M. Menteshashvili. "Some properties of Hadamard matrices". *Computer Science and Information Technologies (CSIT). Revised Selected Papers, September 28 – October 2, 2015, Yerevan, Armenia, IEEE,* pp. 64–66, 2015.

[5] G. Giorgobiani, V. Kvaratskhelia, M. Menteshashvili. "On Some Applications of Hadamard Matrices". *South Caucasus Grid&Cloud Computing Workshop (SCCTW 2016). Tbilisi, Georgian Technical University, 3 – 7 October,* 2016.

[6] W. Hoeffding. "Probability inequalities for sums of bounded random variables". *Journal of the American Statistical Association.* 58 (301), pp.13–30, 1963.

[7] V. Kvaratskhelia. "Some inequalities related to Hadamard matrices". *Functional Analysis and Its Applications*, 36, pp. 81–85, 2002.

[8] J. Spencer. "Balancing vectors in the maximum norm". *Combinatorica*, 6 (1), pp. 55 – 65, 1986.

[9] N.N. Vakhania, V.I. Tarieladze, S.A. Chobanyan. "Probability distributions in Banach Spaces". *D. Reidel Publishing Company, Dordreht-Boston-Lancaster-Tokyo*, 1987.