On the Achromatic Index of Complete Graphs

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ABSTRACT

A *proper edge-coloring* of a graph G is a mapping $\alpha : E(G) \to \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. A proper edge-coloring of a graph G with colors $1, \dots, t$ is called a *complete t-edge-coloring* if for every pair of colors i and j, there are two edges with a common vertex, one colored by i and the other colored by j. The largest value of t for which G has a complete t-edge-coloring is called the *achromatic index* $\psi'(G)$ of G. In this paper we study the achromatic index of complete and complete bipartite graphs. In particular, we prove that for any $m, n \in \mathbb{N}$, $\psi'(K_{m+n+1}) \geq \psi'(K_{m,n})+m+n-1$. We also prove that for any $m, n \in \mathbb{N}$, $\psi'(K_{m,n}) \geq \psi'\left(K_{\frac{m}{(m,n)},\frac{n}{(m,n)}}\right)\left(\psi'\left(K_{(m,n)}\right)+1\right)$, where (m, n) is the greatest common divisor of m and n.

Keywords

Achromatic number, achromatic index, complete edge-coloring, complete graph, complete bipartite graph.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The maximum degree of vertices in G is denoted by $\Delta(G)$, the chromatic number of G by $\chi(G)$ and the chromatic index of G by $\chi'(G)$. We use the standard notations K_n and $K_{m,n}$ for the complete graph on n vertices and the complete bipartite graph, one part of which has m vertices and the other part has n vertices, respectively. For a graph G, by $L(G)$ we denote the line graph of the graph G. The terms and concepts that we do not define can be found in [3, 7, 17].

A proper t-vertex-coloring of a graph G is a mapping $\alpha : V(G) \to \{1, \dots, t\}$ such that for any $uv \in E(G)$, $\alpha(u) \neq \alpha(v)$. The chromatic number $\chi(G)$ of a graph G is the smallest value of t for which it has a proper t-vertex-coloring. A proper t-vertex-coloring of a graph G is a *complete t-vertex-coloring* of a graph G if for every pair of colors i and j, there is an edge uv such that $\alpha(u) = i$ and $\alpha(v) = j$. The *achromatic number* $\psi(G)$ of G is the largest value of t for which G has a complete t-vertex-coloring. The achromatic number of graphs was introduced by Harary and Hedetniemi in [8]. In [9], Harary, Hedetniemi and Prins showed that for any graph G if $\chi(G) \leq t \leq \psi(G)$, then G has a complete t-vertex-coloring. In general, it is known that the problem of determining of the achromatic number is NP-complete for bipartite graphs, cographs, interval graphs, and even for trees [1, 6, 15]. The achromatic numbers of graph operations were considered by Hell and Miller in [10], where the authors obtained some lower bounds for the achromatic number of direct products of graphs.

A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \to \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. A proper edge-coloring of a graph G with colors $1, \dots, t$ is called a *complete t-edge-coloring* if for every pair of colors i and j, there are two edges with a common vertex, one colored by i and the other colored by j. The largest value of t for which G has a complete t-edge-coloring is called the *achromatic index* $\psi'(G)$ of G. Clearly, for any graph G, $\psi'(G) = \psi(L(G))$. The problem of determination of the achromatic index of the complete graph K_n was first considered by Bouchet [2], who proved that there is an intimate connection between this parameter and the existence of finite projective planes.

Theorem 1. *If q is odd and $n = q^2 + q + 1$, then $\psi'(K_n) = q \cdot n$ if and only if a finite projective plane of order q exists. Moreover, if $\psi'(K_n) = q \cdot n$, then the vertices covered by each color class in any complete $\psi'(K_n)$-edge-coloring form the lines of a finite projective plane with the vertices of K_n as points.*

The achromatic index of complete graphs was also considered by Jamison [14]. In [14], the author obtained some lower and upper bounds for the achromatic index of complete graphs. He also showed that $if n > 4$, then $\psi'(K_{n+2}) \geq \psi'(K_n) + 2$. Moreover, Jamison [14] showed that the achromatic index of complete graphs $\psi'(K_n)$ grows asymptotically like $n^{\frac{3}{2}}$. The achromatic index of complete bipartite graphs was first considered by Chiang and Fu [4]. In [4], the authors proved that for any $m, n \in \mathbb{N}$, the following upper bound holds: $\psi'(K_{m,n}) \leq \max_{1 \leq k \leq m} \min\{\lfloor \frac{mn}{k} \rfloor, k(m+n-1)-k^2+1\}$. In [4], Chiang and Fu also proved the following lower bound for $\psi'(K_{m,n})$.

Theorem 2. *For any positive integers $m, n \geq 2$,*

$$\psi'(K_{m,n}) \geq \begin{cases} m+n-1, \text{ if } n > m = 2 \\ \qquad\qquad \text{ or } m = n > 2, \\ 2n - \left\lceil \frac{n}{m-1} \right\rceil, \text{ if } n > m > 2. \end{cases}$$

In the same paper it was proved that $\psi'(K_{2,n}) = n+1$ if $n \geq 3$, and $\psi'(K_{3,3}) = 5$, $\psi'(K_{3,n}) = \left\lfloor \frac{3n}{2} \right\rfloor$ if $n \geq 4$. In [11, 12, 13], the achromatic indices $\psi'(K_{4,n})$ and $\psi'(K_{5,n})$ were determined. In general, the achromatic indices of complete and complete bipartite graphs are unknown. Some other results on the topic were obtained in [2, 5, 14]. In [16], the achromatic indices of graph products were considered. In particular, the authors proved that for any $m, n \in \mathbb{N}$, the following lower bound holds: $\psi'(K_{m \cdot n}) \geq \psi'(K_m) + \psi'(K_n) + \psi'(K_m) \cdot \psi'(K_n)$.

In the present paper we study the achromatic index of complete and complete bipartite graphs. In particular, we prove that for any $m, n \in \mathbb{N}$, $\psi'(K_{m+n+1}) \geq \psi'(K_{m,n}) + m + n - 1$. We also prove that for any $m, n \in \mathbb{N}$, $\psi'(K_{m,n}) \geq \psi'\left(K_{\frac{m}{(m,n)}, \frac{n}{(m,n)}}\right)\left(\psi'\left(K_{(m,n)}\right)+1\right)$.

2. THE MAIN RESULTS

We first prove the following lemma.

Lemma 3. *If for a graph G, $\psi'(G) \geq k \cdot \Delta(L(G))$, then for any complete $\psi'(G)$-edge-coloring of G, each color is used at least $k-1$ times.*

Using this lemma we prove the following result on the achromatic index of complete graphs.

Theorem 4. *For any $m, n \in \mathbb{N}$, we have*

$$\psi'(K_{m+n+1}) \geq \psi'(K_{m,n}) + m + n - 1.$$

Next we show that there is a connection between the achromatic indices of complete and complete bipartite graphs.

Theorem 5. *For any $n \in \mathbb{N}$, we have*

$$\psi'(K_{n,n}) \geq \psi'(K_n) + 1.$$

Proof. Let $V(K_n) = \{v_1, \ldots, v_n\}$ and α be a complete $\psi'(K_n)$-edge-coloring of K_n. Also, let $V(K_{n,n}) = U \cup W$, where $U = \{u_1, \ldots, u_n\}$, $W = \{w_1, \ldots, w_n\}$ and $E(K_{n,n}) = \{u_i w_j \colon 1 \leq i \leq n, 1 \leq j \leq n\}$.

Define an edge-coloring β of $K_{n,n}$ as follows:

1) for every edge $v_i v_j \in E(K_n)$, let

$$\beta(u_i w_j) = \beta(u_j w_i) = \alpha(v_i v_j);$$

2) for $i = 1, \ldots, n$, let

$$\beta(u_i w_i) = \psi'(K_n) + 1.$$

It is not difficult to see that β is a complete $(\psi'(K_n)+1)$-edge-coloring of $K_{n,n}$. Thus, $\psi'(K_{n,n}) \geq \psi'(K_n) + 1$. $\square$

Using the previous theorem we prove the following results on the achromatic index of complete bipartite graphs.

Theorem 6. *For any $m, n \in \mathbb{N}$, we have* $\psi'(K_{m,n}) \geq \psi'\left(K_{\frac{m}{(m,n)}, \frac{n}{(m,n)}}\right)\left(\psi'\left(K_{(m,n)}\right)+1\right)$.

Theorem 7. *For any $n \in \mathbb{N}$, we have*

$$\psi'(K_{n,n}) \geq \max_{d|n (d \neq 1)} \left\{\psi'\left(K_{\frac{n}{d}, \frac{n}{d}}\right)\left(\psi'(K_d)+1\right)\right\}.$$

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