On the Existence of Bipartite Graphs Which are not Cyclically-Interval Colorable

Rafayel R. Kamalian

Institute for Informatics and Automation Problems,
National Academy of Sciences of Armenia,
0014, Yerevan, Armenia

e-mail: rrkamalian@yahoo.com

ABSTRACT

A proper edge t-coloring of an undirected, simple, finite, connected graph G is a coloring of its edges with colors $1, 2, ..., t$ such that all colors are used, and no two adjacent edges receive the same color. A cyclically interval t-coloring of a graph G is a proper edge t-coloring of G such that for each its vertex x, either the set of colors used on edges incident to x or the set of colors not used on edges incident to x forms an interval of integers. For any $t \in N$, let $\mathfrak{M}_t$ be the set of graphs for which there exists a cyclically-interval t-coloring. Examples of bipartite graphs that do not belong to the class $\bigcup_{t\geq 1} \mathfrak{M}_t$ are constructed.

Keywords

Cyclically interval t-coloring, bipartite graph.

We consider undirected, simple, finite, and connected graphs. For a graph G we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For a graph G, we denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree of a vertex of G and the chromatic index of G [1], respectively. The terms and concepts which are not defined can be found in [2].

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A.

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Cyclically-interval colorings of bipartite graphs can be used for the mathematical modelling of the problems of existence and construction of time-tables of "two-sided" processes. An example of this is the production, in which one side represents the requirements, and the other side represents the machines which meet these requirements. Cyclically-interval colorings correspond to the case when the process should be organized without "gaps" in a "24-hour" regime. In such a case, the color of the edge in a cyclically-interval coloring of a bipartite graph is interpreted as the number of the "hour", during which the machine meets the requirement.

For any $t \in N$, we denote by $\mathfrak{M}_t$ the set of graphs for which there exists a cyclically-interval t-coloring. Let

$$\mathfrak{M} \equiv \bigcup_{t\geq 1} \mathfrak{M}_t.$$

For an arbitrary tree D, it was shown in [3, 4] that $D \in \mathfrak{M}$, and, moreover, all possible values of t were found for which $D \in \mathfrak{M}_t$. For an arbitrary simple cycle C, it was shown in [5, 6] that $C \in \mathfrak{M}$, and, moreover, all possible values of t were found for which $C \in \mathfrak{M}_t$. Some interesting results on this and related topics were obtained in [7–21].

In this paper, the examples of bipartite graphs that do not belong to the class $\mathfrak{M}$ are constructed.

For any integer $m \geq 2$, set:

$$V_{0,m} \equiv \{x_0\}, V_{1,m} \equiv \{x_{i,j}/1 \leq i < j \leq m\},$$

$$V_{2,m} \equiv \{y_{p,q}/1 \leq p \leq m, 1 \leq q \leq m\},$$

$$E'_m \equiv \{(x_0, y_{p,q})/1 \leq p \leq m, 1 \leq q \leq m\}.$$

For any integers i, j, m satisfying the inequalities $m \geq 2$, $1 \leq i < j \leq m$, set:

$$E''_{i,j,m} \equiv \{(x_{i,j}, y_{i,q}), (x_{i,j}, y_{j,q})/1 \leq q \leq m\}.$$

For any integer $m \geq 2$, let us define a graph $G(m)$ by the following way:

$$G(m) \equiv \left(\bigcup_{k=0}^{2} V_{k,m}, E'_m \cup \left(\bigcup_{1\leq i<j\leq m} E''_{i,j,m} \right) \right).$$

It is not difficult to see that for any integer $m \geq 2$, $G(m)$ is a bipartite graph with $\Delta(G(m)) = \chi'(G(m)) = m^2$, $|V(G(m))| = \frac{3m^2-m}{2} + 1$, $|E(G(m))| = m^3$.

Theorem 1. For any integer $m \geq 8$, $G(m) \notin \mathfrak{M}$.

We prove the theorem by the method of contradiction. We show that for any $m \geq 8$, if the statement of the theorem is wrong, then in every cyclically-interval t-coloring of $G(m)$ $(\Delta(G(m)) \leq t \leq |E(G(m))|)$ we can find an edge (z', z'') of the graph, the color of which cannot be used for an edge that is incident to z'.

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