On Complete Caps in Galois Affine Space $AG(n, 3)$

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ABSTRACT

A cap in a projective or affine space over a finite field F_q with q elements is a set of points no three of which are collinear. We give two new recurrence constructions for complete caps in affine space $AG(n, 3)$, which leads to some new upper and lower bounds on the possible minimal and maximal cardinality of complete caps, respectively.

Keywords

Affine space, projective space, P_n set, cap, complete cap.

1. INTRODUCTION

A cap in a projective $PG(n,q)$ space or affine $AG(n,q)$ space over a finite field F_q with q elements is a set of points (vectors) no three of which are collinear. A cap is called complete when it cannot be extended to a large one. The main problem in the theory of caps is to find the minimal and maximal sizes of complete caps in $PG(n,q)$ or $AG(n,q)$, see the survey papers [1, 2, 3] and the references therein. Note that the problem of determining the minimum size of a complete cap in a given space is of particular interest in Coding Theory [2]. The only complete cap in $AG(n,2)$ is the whole $AG(n,2)$. The trivial lower bound for the size of the smallest complete cap in $AG(n,q)$ is $\sqrt{2} \cdot q^{\frac{n-1}{2}}$. For q even, there exist complete caps in $AG(n,q)$ with less than $q^{\frac{n}{2}}$ points [4, 5, 6]. But for q odd, complete caps in $AG(n,q)$ with less than $q^{\frac{n}{2}}$ points are known only for $n \equiv 0(mod 4)$, $n \equiv 2(mod 4)$ and for small values of n and q [3, 6, 7, 8]. In this paper we give two new recurrence constructions for complete caps in affine space $AG(n,3)$.

2. MAIN RESULTS

It is easy to see that if S is a cap in $AG(n,3)$, then $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} \neq \mathbf{0}$ $(mod 3)$ for any triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in S$.

As in [9], let's denote by $B_n = \{(\alpha_1, \dots, \alpha_n) / \alpha_i = 0{,}1\}$ and by P_n the set of points of $AG(n,3)$ satisfying the following two conditions:

- i) for any triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in P_n$, $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} \neq \mathbf{0}(mod 3)$,
- ii) for any two distinct points $\boldsymbol{\alpha},\boldsymbol{\beta} \in P_n$, there exists i $(1 \le i \le n)$ such that $\alpha_i = \beta_i = 2$.

We call P_n to be complete when it cannot be extended to a larger one.

We will define the concatenation of the points in the following way. Let $A \subset AG(n,3)$ and $B \subset AG(m,3)$. We form a new set $AB \subset AG(n+m,3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m})$, where $\boldsymbol{\alpha}' = (\alpha_1, \dots, \alpha_n) \in A$ and $\boldsymbol{\alpha}'' = (\alpha_{n+1}, \dots, \alpha_{n+m}) \in B$. In a similar way one can define the concatenation of the points of three sets, four sets,…etc. Note that if $x,y,z \in F_3$, then $x+y+z \equiv 0\,(mod 3)$ if and only if $x=y=z$ or they are pairwise distinct.

It is obvious that $P_1 = \{2\}$ and $P_2 = \{(2,0),(2,1)\}$ or $P_2 = \{(0,2),(1,2)\}$ and they are complete. Presenting the natural numbers as the sum of three (six) natural numbers and applying Theorem 1 (Theorem 2), one can obtain complete P_n sets for each n.

Theorem 1. The following recurrence relation $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, with initial sets $P_1 = \{2\}, P_2 = \{(2,0),(2,1)\}$ or $P_2 = \{(0,2),(1,2)\}$ and $n = n_1 + n_2 + n_3$, yields complete sets.

Let's form the following ten sets:

$A_1 = P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6}$, $A_2 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6}$
$A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}$, $A_4 = B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}$
$A_5 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_6 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$A_7 = B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_8 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}$, $A_{10} = P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}$.

Theorem 2. The following recurrence relation $P_n = \bigcup_{i=1}^{10} A_i$, with initial sets $P_1 = \{2\}, P_2 = \{(2,0),(2,1)\}$ or $P_2 = \{(0,2),(1,2)\}$ and $n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$, yields complete P_n sets.

Note that the cardinality of P_n, obtained by Theorem 1 (Theorem 2), essentially depends on the representation of n as the sum of three (six) natural numbers. Presenting the natural numbers as the sum of six natural numbers and applying Theorem 2, for some $n \ge 6$ one can obtain larger complete P_n sets than those, which are constructed by Theorem 1.

Theorem 3. If P_n and P_m are complete sets constructed by Theorem 1 or Theorem 2, then $P_nB_m \cup B_nP_m$ is a complete cap.

Theorem 4. If P_i and P_{n-i} are complete sets $(1 \le i \le n-1)$ constructed by Theorem 1 or Theorem 2, then $P_iP_{n-i} \cup P_iB_{n-i} \cup B_iP_{n-i} \cup B_n$ is a complete cap.

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