On the Size of Maximum k-Edge-Colorable Subgraphs in Regular Graphs

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ABSTRACT

An edge-coloring of a graph is an assignment of colors to edges of the graph, so that adjacent edges receive different colors. A subgraph H of a graph G is maximum k-edge-colorable, if H is k-edge-colorable and contains as many edges as possible. In this paper, we present some results towards the problem of finding a tight lower bound for $\frac{|E(H)|}{|E(G)|}$, where H is a maximum k-edge-colorable subgraph of G, and G is an arbitrary or an regular graph.

Keywords

Edge coloring, k-edge-coloring, k-edge colorable subgraph, r-regular graph

1. INTRODUCTION

The classical edge coloring problem is frequently used to model real-world problems of resource allocation, such as scheduling of tasks requiring the cooperation of two processors, file transfer operations, and assignment of channels of satellite communication.

For any positive integer k there are graphs that do not admit an edge-coloring with k colors. Thus, an interesting problem is to find the maximum number of edges we can color the given k colors.

In this paper, we consider finite, undirected graphs with no loops. Graphs may contain parallel edges. For a graph G, let $V(G)$ denote its vertex set and $E(G)$ its edge set. The degree of a vertex v of G, denoted by $d_G(v)$, is the number of edges of G that are incident to v. A graph is r-regular if $d_G(v) = r$ for all $v \in V(G)$. The maximum of $d_G(v)$ over all $v \in V(G)$ is called a maximum degree of G and is denoted by $\Delta(G)$. For a vertex v of G let $\partial_G(v)$ be the set of edges of G incident to v. The girth of a graph is the length of a shortest cycle of the graph.

A matching M in G is a set of pairwise non-adjacent edges, that is, a subset of edges, where no two edges share a common vertex. A k-factor of a graph G is a spanning k-regular subgraph of G. Thus, the edge-set of a 1-factor is a perfect matching.

A k-edge coloring of a graph G is an assignment of k colors to the edges of G, so that adjacent edges receive different colors. A k-edge coloring can be thought of as a partition $(E_1, E_2, ..., E_k)$ of $E(G)$, where E_i denotes the subset of $E(G)$ having color i. It is not hard to see that a k-edge coloring is just a partition $(E_1, E_2, ..., E_k)$, in which each subset E_i is a matching [9].

The least number k for which G has a k-edge coloring is called a chromatic index of G and is denoted by $\chi'(G)$. The graphs G with $\chi'(G) = \Delta(G)$ are said to be class I, otherwise they are class II. Shannon's theorem says that for any G graph $\Delta(G) \leq \chi'(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor$ and Vizing's theorem states that for any G graph $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ where the $\mu(G)$ is the maximum multiplicity of an edge in G.

For a graph G and a positive integer k, a subgraph H of G is called maximum k-edge colorable if it is k-edge-colorable and contains as many edges as possible. The number of edges of H is denoted by $\nu_k(G)$. The maximum k-edge-coloring subgraph problem is the following: for a given graph G find a maximum size subgraph H of G, which is k-edge-colorable. In other words, in this problem we are looking for a k-edge-colorable subgraph containing $\nu_k(G)$ edges.

There are several papers where the ratio $\frac{|E(H)|}{|E(G)|}$ has been investigated. In [8], an algorithm for the problem is presented. There for each fixed value of $k \geq 2$, a polynomial time approximation algorithm is described, where the approximation ratios are tending to 1 as k tends to infinity. In [1], it is shown that any 2-factor of a cubic graph can be extended to a maximum 3-edge colorable subgraph. Also the authors proved that for every cubic graph G

$$\nu_2(G) \geq \frac{4}{5}|V(G)| \text{ and } \nu_3(G) \geq \frac{7}{6}|V(G)|.$$

Moreover, it can be shown that

$$\nu_2(G) + \nu_3(G) \geq 2|V(G)|,$$

and

$$\nu_2(G) \leq \frac{|V(G)| + 2\nu_3(G)}{4}$$

The last equality has been investigated in [5]. There, it is shown that

$$\nu_2(G) \geq \alpha \cdot \frac{|V(G)| + 2\nu_3(G)}{4}$$

where $\alpha = \frac{16}{17}$, if G is a cubic graph, $\alpha = \frac{20}{21}$ if G is a cubic graph containing a perfect matching and $\alpha = \frac{44}{45}$ if G is a bridgeless cubic graph. There, also the improved lower bounds of $\nu_2(G)$ and $\nu_3(G)$ are proved when G is a claw-free bridgeless cubic graph:

$$\nu_2(G) \geq \frac{35}{36} \cdot |V(G)|, \nu_3(G) \geq \frac{43}{45} \cdot |E(G)|.$$

There are some results in [10] about maximum Δ-edge colorable subgraph of class II graphs. There, the authors proved that every set of disjoint cycles of a class II graph with $\Delta \geq 3$ can be extended to a maximum Δ-edge colorable subgraph. It is also shown there that a maximum Δ-edge colorable subgraph of a simple graph is always class I. Finally, if G is a graph with girth $g \in \{2k, 2k+1\}$ $(k \geq 1)$ and H is a maximum Δ-edge colorable subgraph of G, then

$$\frac{|E(H)|}{|E(G)|} \geq \frac{2k}{2k+1}$$

and the bound is best possible in a sense that there is an example attaining it.

Finally, let us note that the lower bounds for $\frac{\nu_k(G)}{|V(G)|}$ in cubic graphs have been investigated in [2, 6, 11, 12, 14] when $k = 1$, and for regular graphs of high girth in [3]. These lower bounds have also been investigated in the case when the graphs need not be cubic [4, 7, 13].

In this paper, we prove a best-possible bound for $\frac{|E(H)|}{|E(G)|}$ in the class of all graphs. We also investigate the same problem in the class of regular graphs and present some partial results in relation to them. Non-defined terms and concepts can be found in [9].

2. RESULTS

In this section, we present our main results. First, we consider the following problem in the class of all graphs.

Problem 1. *For $\Delta \geq 1$ and $k = 1, ..., \left[\frac{3\Delta}{2}\right]$ define the function $g(\Delta, k)$ as the infimum of $\frac{|E(H_k)|}{|E(G)|}$, where G is any graph and H_k is a maximum k-edge-colorable subgraph of G. The infimum is taken over all graphs G of maximum degree Δ. The problem is to determine the function g.*

Our first result states:

Theorem 1. *For $\Delta \geq 1$ and $k = 1, ..., \left[\frac{3\Delta}{2}\right]$, we have $g(\Delta, k) = \frac{k}{\left[\frac{3\Delta}{2}\right]}$.*

Proof. First let us show that $g(\Delta, k) \leq \frac{k}{\left[\frac{3\Delta}{2}\right]}$. Consider a graph G on three vertices a, b and c, where a and b are joined with $\left\lfloor\frac{\Delta}{2}\right\rfloor$ edges, a and c are joined with $\left\lfloor\frac{\Delta}{2}\right\rfloor$ edges, and b and c are joined with $\left\lceil\frac{\Delta}{2}\right\rceil$ edges. It can be easily seen that $\frac{|E(H_k)|}{|E(G)|} = \frac{k}{\left[\frac{3\Delta}{2}\right]}$.

In order to prove the opposite inequality, let G be any graph with maximum degree Δ. From Shannon theorem we have $\chi'(G) \leq [\frac{3\Delta}{2}]$, thus, to prove the theorem it's enough to prove that $\nu_k(G) \geq \frac{k}{\chi'(G)} \cdot |E(G)|$. It will be proved using the following proposition:

Proposition 1. *Let $a_1 \geq ... \geq a_n$ be any positive numbers and let $k \leq n$. Then the arithmetical mean of the $a_1, ..., a_k$ is not less than the arithmetical mean of $a_1, ..., a_n$.*

Consider an edge-coloring of G with $n = \chi'(G)$ colors. Let $a_1, ..., a_n$ be the sizes of the color classes. We can assume that $a_1 \geq ... \geq a_n$. By Proposition 1, we have

$$\frac{\nu_k(G)}{k} \geq \frac{a_1 + ... + a_k}{k} \geq \frac{a_1 + ... + a_n}{n} = \frac{|E(G)|}{\chi'(G)},$$

which proves the theorem. □

The second problem that we considered is the following:

Problem 2. *For $r \geq 3$ and $k = 1, ..., \left[\frac{3r}{2}\right]$ define the function $f(r, k)$ as the infimum of $\frac{|E(H_k)|}{|E(G)|}$, where G is any r-regular graph and H_k is a maximum k-edge-colorable subgraph of G. The infimum is taken over all r-regular graphs G. The problem is to determine the function f.*

We suspect that:

Conjecture 1. *For $r \geq 3$ and $k = 1, ..., \left[\frac{3r}{2}\right]$ one has:*

$$f(r,k) = \begin{cases} \frac{2k}{3r}, & \text{if } r \text{ is even;} \\ \frac{2k(r+1)}{r(3r+1)}, & \text{if } r \text{ is odd and } k \leq \frac{3r+1}{4}; \\ \frac{2k+1}{3r}, & \text{if } r \text{ is odd and } k \geq \frac{3r+1}{4}. \end{cases}$$

Note that if r is odd and $k = \frac{3r+1}{4}$, then the two expressions give the same value.

Related with this conjecture, we are able to show:

Theorem 2. *Conjecture 1 is true when r is even, or when r is odd and $k = 1$.*

Proof. We start with the case when r is even. By Theorem 1, we have $f(r, k) \geq \frac{2k}{3r}$. On the other hand, if we consider the same example from the proof of Theorem 1 when $\Delta = r$, one can easily see that we get an r-regular graph, hence the converse inequality is also true. Thus, $f(r, k) = \frac{2k}{3r}$ when r is even.

Now, let us consider the case when r is odd and $k = 1$. Consider the following graphs A_r and B_r. We make A_r by taking r copies of the Shannon's triangle (the graph from the proof of Theorem 1 when $\Delta = r$), and taking a new vertex z and joining it to r vertices that are of degree $r - 1$, thus we get an r-regular graph and B_r by taking two copies of the Shannon's triangle, and joining the two vertices of degree $r - 1$ with an edge. Again we get an r-regular graph. Now, it is a matter of direct verification, that if $k \leq \frac{3r+1}{4}$ then A_r attains the bound of the conjecture, while for $k \geq \frac{3r+1}{4}$, B_r attains the bound of the conjecture. Thus, $f(r, k)$ is at most the bound predicted by the Conjecture 1, in particular, when $k = 1$.

Now, let us show that $f(r, 1)$ is at least the bound of the Conjecture 1 when $k = 1$. Nishizeki in [11] has shown that any odd r-regular graph G contains a matching of size at least $\left\lceil\frac{(r^2-r-1)|V|-(r-1)}{r(3r-5)}\right\rceil$. Now it can be shown that this expression is at least $\frac{r+1}{3r+1} \cdot |V(G)| = \frac{2(r+1)}{r(3r+1)} \cdot |E(G)|$. Thus, $f(r, 1) \geq \frac{2(r+1)}{r(3r+1)}$. □

Finally, it turns out that there is a certain dependence among different values of k in the Conjecture 1. More precisely, we are able to show that

Theorem 3. *If Conjecture 1 is true when r is odd and* $k = \lfloor \frac{3r+1}{4} \rfloor$*, then it is true when r is odd and* $k = 1, ..., \lfloor \frac{3r+1}{4} \rfloor$.

Proof. Assume that r is odd. Observe that for $i = 1, ..., k$, where $k = \lfloor \frac{3r+1}{4} \rfloor$, we have $\nu_i(G) \geq \frac{i}{k} \cdot \nu_k(G)$. Hence, if Conjecture 1 is true when $k = \lfloor \frac{3r+1}{4} \rfloor$ then it is true for $i = 1, ..., k$. $\square$

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