Construction Method of (2N, 2N – 6) Linear Codes over Ring Z_m, Based on (N, N – 4) Linear Codes Correcting Double ±1 Types of Errors

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ABSTRACT
From a practical point of view the codes over rings Z_{2m} or Z_{2m+1} are interesting, because they can be used in QAM **(**Quadrature amplitude modulation) schemes**.** Until today we only know codes with 4 parity check symbols, which have a limited length C (N, N-4). In this paper a method allowing to construct C(2N, 2N-6) double ±1 error correcting codes over rings Z_m from the given C(N, N-4) double ±1 error correcting codes over rings Z_m is developed by adding only two extra parity check symbols.

Keywords
Error correcting codes, asymmetrical errors, optimal codes.

1. INTRODUCTION
Errors happening in the channel are basically asymmetrical; they also have a limited magnitude and this effect is particularly applicable to flash memories. There have been a couple of papers regarding optimal *±1* single error correcting codes over alphabet Z_m [1, 2]**.** Also there are many linear codes capable to correct up to two errors of type *±1* for different alphabets which have been found by computer search, but they are not optimal. The optimality criteria for linear codes over the fixed ring Z_m can be considered in two ways. First of all, recall that the code of the length n is optimal-1 if it has a minimum possible number of parity check symbols. Secondly, optimality-2 criteria for the code is that for a given number of parity check symbols, it has a maximum possible length. The linear code (12, 8) correcting double errors over the ring Z_5 of value ±*1* was presented in [3], satisfies the optimality criteria -1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 3 & 2 & 4 & 4 & 2 & 3 & 2 & 4 & 4 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 2 & 4 & 4 & 2 & 0 & 4 \end{bmatrix}$$

this code was given by the parity check matrix *H*, which has 8 information and 4 parity check symbols. Other optimal codes over the rings Z_7 and Z_9 correcting double errors of value ±*1* was presented in [4]. There are code (16, 12) over the ring Z_7,

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 4 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 1 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 0 & 0 \end{bmatrix}.$$

and code (20, 16) over the ring Z_9:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 2 & 4 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 4 \\ 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 6 & 3 & 7 & 2 \end{bmatrix}$$

At this point we do not know any codes that satisfy the optimality criteria-2. In [3] a method is presented how to compare two code constructions over different size of alphabets when both satisfy the optimality–1 criteria. Two factors are considered, namely:

- the first factor should be the rate of the code i.e., the ratio of the number of information symbols over the length of the code;
- the second factor should be the ratio between the number of possible amplitude errors corrected by the code over the size of alphabet minus 1, which corresponds to the number of all possible amplitude errors.

For the code over ring Z_5 mentioned above the product is: (8/12) * (2/4) = 0.3333. For the codes over the rings Z_7 and Z_9 the products will be (12/16) * (2/6) = 0.25 for the code over Z_7 and (16/20) * (2/8) = 0.2 for Z_9. These products are a little bit smaller than for the code (12, 8) over ring Z_5, although there are much better comparisons with the codes over Z_{16} and Z_{128} in [2,5,6].

In this paper constructions of codes C (2N, 2N-6) are presented, which are based on the previous optimal codes C (N, N-4). We can construct codes which are 2 times longer than C (N, N-4) optimal codes, by adding only 2 parity check symbols.

2. CONSTRUCTION OF C(2N,2N-6) CODES
In this section we will describe a method, which allows us to construct new codes with a double length at the expense of just two parity check symbols. We will assume that we have at our disposal (N, N-4) double error correcting code like the codes presented in Paragraph 1 in this paper as well as in [3,4]. Using the method, which will be described below, we can construct codes of length 2N with 6 parity check symbols C (2N, 2N-6). Further in this paper by codes we mean a double error correcting code of the type ±*1*.

Let C (N, N-4) be a code over ring Z_m. Our construction of a new code will have a parity check matrix with 6 rows and a 2N columns where the first 4 rows will be just a repetition of 4 rows of the code C (N, N-4). Now we will describe how we add 2 additional rows to the parity check matrix. The first N columns and the last N columns of two additional rows will be referred to as group 1 and a group 2, respectively. In the first row of group 1 we put integer 2 repeated n times then integer 1 repeated n times and then (N - 2n) times any integer x from Z_n. In the second row of the group 1 we put all integers from Z_n {0, 1 ,…, n-1} repeated twice and then the first (N-2n) elements of {0,1,…, n-1}. Consequently, in the first row of group 2 we put the second row of group 1, and in the second row of group 2 accordingly we put integers 3 and 4 repeated n times, and in the rest of positions the same integer x from group 1. Note that an integer x should differ from (1, 2, 3, 4) and must satisfy the condition (($x + x \neq 1 \ (mod\ n)$)),

Now we can prove the following theorem.

Theorem 1: *For a given C (N, N-4) code over ring* Z_n *correcting double ±1 errors, it is possible to construct a code with 6 parity check symbols of a length C (2N, 2N-6) correcting ±1 double errors.*

Proof. In order to prove this theorem it must be shown that all corresponding syndromes resulting from operations ±*1* between all columns of both groups should be all different.

Group 1 C (N, N - 4)	Group 2 C (N, N - 4)

$$\begin{bmatrix} 2 & 2 & \dots & 2 & 1 & 1 & \dots & 1 & x & x & \dots & 0 & 1 & \dots & n-1 & 0 & 1 & \dots & n-1 & 0 & 1 & \dots \\ 0 & 1 & \dots & n-1 & 0 & 1 & \dots & n-1 & 0 & 1 & \cdots & 3 & 3 & \dots & 3 & 4 & 4 & \dots & 4 & x & x & \dots \end{bmatrix}$$

Let us split all columns of the group 1 into 3 subgroups, namely the first subgroup (subgroup 1.1) contains the first columns of group 1, the first component of which is 2, starting from the left, the second subgroup (subgroup 1.2) contains next n columns, the first component of which is 1, and the third subgroup (subgroup 1.3) contains the last columns with x. Accordingly, we can do the same with columns of group 2 and split them into three subgroups (2.1, 2.2 and 2.3) . We need to consider only those cases, when the first four components have the same syndromes. We will demonstrate the proof for the case, when the first error has an upward direction (+1), and the second - downward (-1). Let us perform the proof by 3 cases:

1) Let's suppose that two errors occurred in the first group. Because the both parts of parity check matrix H consist of the same code in the first 4 rows, we do not know in which part the errors occur: whether in the first part of matrix H or in the second one. We can check it using the next 2 rows. There can be only 3 possible subcases:

1.1) If errors are in subgroup 1.1 the first position will always be 0, otherwise in group 2 it cannot be 0 (due to the property of the set {0,1,…, n-1}), and the syndromes will be different.

1.2) If one error occurs in subgroup 1.1 and the second in subgroup 1.2 in group 1 the first position will be 1, but in group 2 the second position will be -1, and the syndromes will always be different, because we have the same components in two other positions (the second row of group 1 and the first row of group 2 are the same).

1.3) If one of the errors occurs in subgroup 1.3, then if next is in subgroup 1.1 resulting syndromes will be different, because in subgroup 1.1 the first position is 2 and in subgroup 2.1 the second position is 3, but we have the same components as in subgroups 1.3 and 2.3. Like the case b) the other two components of the syndrome always will be the same (the second row of group 1 and the first row of group 2 are the same). If the second error occurs in subgroup 1.2(2.2) the way of the proof is the same. Thus, the first case of the proof is complete.

2) Let one error occur in group 1 and the second in group 2. We need to check whether both errors are in the same group or not. Again there can be 3 possible subcases:

2.1) Let an error occur in subgroup 1.1 and second error in subgroup 2.1. As the first four components of matrix H are in these subgroups we have the same columns, the errors might be in the same subgroup. How can we distinguish between these cases? If both of them occur in subgroup 1.1, then the first component will be 0, otherwise, in our case 0 can be only with column 3 of subgroup 2.1. In this case, in the third column of subgroup 1.1 the second position is 2, and in the same column of subgroup 2.1 it is 3, consequently the syndromes will be different (if both of them occur in subgroup 2.1 the proof is the same(the only difference is that the second component will be 0).

2.2) Let one error occur in subgroup 1.1 and the second in subgroup 2.2. If both errors occur in group 1, then the first component will be 1, in our case 1 can be only with column 2 of subgroup 2.2, here the second component is 4, but in subgroup 1.2 it is 1, consequently the syndromes will be different. If both of them occur in group 2, then the second component will be -1, in our case -1 can be only with column 4 of subgroup 1.1, here the first component is 2, but in subgroup 2.1 it is 3, and syndromes will be different. (For the case when errors occur in subgroup 2.1 and subgroup 1.2 the way of proof is the same).

2.3) Let one error occur in subgroup 1.1 or in subgroup 1.2 and the second in subgroup 2.3. If both errors occur in group 1(subgroup 1.3) the resulting syndromes will be different, because in subgroups 1.3 and 2.3 the corresponding rows are swapped like $\begin{pmatrix} x & x & \dots \\ 0 & 1 & \dots \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & \dots \\ x & x & \dots \end{pmatrix}$. Consequently, when we subtract them from the same subgroup 1.1 or 1.2 the resulting syndromes will be different. (For the case when errors occur in subgroup 1.3 and in subgroups 2.1 or 2.2 the way of proof is the same). Thus, the second case of the proof is complete.

3) In this case both of the errors occur in the same columns of different groups. Accordingly, the first four components of the syndromes will be (0 0 0 0). In this case the number of all possible syndromes will be 2N . Due to the selection of last two rows of matrix (group 1 and group 2), it can be shown that all 2N syndromes will be different. In this case the difference between the same columns for the first two subgroups of groups 1 and 2 will be if the second element of the last column is the same first elements will be different by two, while the difference between the same columns in the corresponding third subgroups will be $\begin{pmatrix} x - i \\ i - x \end{pmatrix}$ and will be different for all i's unless 2x ≠1 (mod **n**) - which is the condition for x. This analysis completes the proof of the theorem.

3. CONCLUSION

In this paper a construction method of C(2N, 2N-6) double ±1 error correcting codes over rings Z_m based on C(N, N-4) double ±1 error correcting codes over rings Z_m is developed. This technique will allow to construct codes with higher rates.

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Here we will show some results, which we have gotten using the method described by Theorem 1:

Example 1

The code (24, 18) correcting double ± 1 errors, given by parity check matrix H_5 was resulted from the optimal code (12, 8) over ring Z_5 by adding 2 parity check symbols:

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 3 & 2 & 4 & 4 & 2 & 3 & 2 & 4 & 4 & 2 & 1 & 1 & 3 & 2 & 4 & 4 & 2 & 3 & 2 & 4 & 4 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 3 & 2 & 4 & 4 & 2 & 0 & 4 & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 4 & 4 & 2 & 0 & 4 \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 0 & 0 \end{bmatrix}.$$

Example 2

The code (32, 26) correcting double ± 1 errors, given by parity check matrix H_7 was resulted from the optimal code (16, 12) over ring Z_7 by adding 2 parity check symbols:

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 \end{bmatrix}$$

$$H_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 1 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 4 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 1 & 6 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 1 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 3 & 6 & 6 & 3 & 4 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 & 5 \end{bmatrix}.$$

Example 3

And lastly, the code (40, 34) correcting double ± 1 errors, given by parity check matrix H_9 was resulted from the optimal code (20, 16) over ring Z_9 by adding 2 parity check symbols:

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 6 & 6 \end{bmatrix}$$

$$H_9 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 2 & 4 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 4 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 4 \\ 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 1 & 1 & 2 & 4 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 6 & 3 & 7 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 3 & 2 & 4 & 4 & 2 & 3 & 7 & 6 & 3 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 6 & 6 \end{bmatrix}.$$