On Construction of Probability Currents Between Asymptotic Subspaces of Multichannel Scattering

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ABSTRACT
In this paper we prove: *The general three-body problem formulated on Riemannian geometry makes it possible to discover new hidden symmetries of the internal motion of a dynamical system and reduce the problem to the system of order 6* th. *It is shown that the equivalence of the initial Newtonian three-body problem and developed representation provides coordinate transformations in combination with an underdefinished system of algebraic equations. The latter makes a system of geodesic equations relative to the evolution parameter, i.e., the arc length of the geodesic curve,* irreversible. *To describe the motion of a dynamical system influenced by external regular and stochastic forces, a system of stochastic equations (SDE) is obtained. Using the system of SDE, a partial differential equation of the second order for the joint probability distribution of the momentum and coordinate of dynamical system in the phase space is obtained. A criterion for estimating the degree of deviation of probabilistic current tubes of geodesic trajectories in the phase and configuration spaces is formulated. The mathematical expectation of the transition probability between two asymptotic channels is determined, taking into account the multichannel character of the scattering.*

Keywords
Classical three-body problem, Riemannian space, irreversibly system, stochastic equations, probability current.

1. INTRODUCTION
In the recent work [1] it is proved that the three-particle classical problem in the most general case is equivalent to the problem of geodesic flows on a Riemannian manifold. As shown, the formulation of the problem on the curved space allows to reveal the hidden internal symmetries of the dynamic system, which helps to achieve a more complete reduction of the problem. Namely, within the framework of the new representation, the three-body problem instead of the system of the 8th order is described by the system of the 6th order.

In this article, we consider the problem of three bodies under the influence of the environment, taking into account that the influence of the environment on the body system has both regular and stochastic impacts. A new type of second-order partial differential equations describing geodesic flows on a Riemannian manifold is derived. It is proved that the timing parameter in this equation branches during the evolution that making the equation irreversible relative to it.

2. FORMULATION OF THE PROBLEM
The classic three-body problem in a most general formulation, is the problem of multichannel scattering, with series of possible asymptotic outcomes, which schematically can be represented in the form:

$$1\ +\ (23) \longrightarrow (123)^* \longrightarrow \begin{array}{l} 1+2+3, \\ 1\ +\ (23), \\ (12)\ +\ 3, \\ (13)\ +\ 2, \\ (123)^{**} \nearrow . \end{array} \qquad (1)$$

Scheme 1: *where 1, 2 and 3 indicate single bodies, the bracket* (.) *denotes the two-body bound state, while* "$*$" *and* "$**$" *denote some of transition states of the three-body system and finally* $\nearrow$ *denote decay of the transition state into one of the asymptotic states.*

Note, that the main goal of the study of the problem is to construct a mathematical expectation of the probabilities of transitions between the indicated on the diagram by asymptotic states.

3. PROBABILISTIC CURRENTS IN THE PHASE SPACE
As shown in [1], the classical three-body problem in the general case reduces to the system of order $6th$:

$$\begin{aligned} &\dot{\xi}^1 = A^1(\{\bar{x}\},\{\bar{\xi}\}), && \xi^1 = \dot{x}^1, \\ &\dot{\xi}^2 = A^2(\{\bar{x}\},\{\bar{\xi}\}), && \xi^2 = \dot{x}^2, \\ &\dot{\xi}^3 = A^3(\{\bar{x}\},\{\bar{\xi}\}), && \xi^3 = \dot{x}^3, \qquad \dot{\xi}^i = d\xi^i/ds, \end{aligned} \qquad (2)$$

where $\{\bar{\xi}\} = (\xi^1,\xi^2,\xi^3)$ and $\{\bar{x}\} = (x^1,x^2,x^3)$, in addition, the following designations are made:

$$\begin{aligned} A^1(\{\bar{x}\},\{\bar{\xi}\}) &= a_1\big\{(\xi^1)^2-(\xi^2)^2-(\xi^3)^2-\Lambda^2\big\} \\ &\qquad +2\xi^1\big\{a_2\xi^2+a_3\xi^3\big\}, \\ A^2(\{\bar{x}\},\{\bar{\xi}\}) &= a_2\big\{(\xi^2)^2-(\xi^3)^2-(\xi^1)^2-\Lambda^2\big\} \\ &\qquad +2\xi^2\big\{a_3\xi^3+a_1\xi^1\big\}, \\ A^3(\{\bar{x}\},\{\bar{\xi}\}) &= a_3\big\{(\xi^3)^2-(\xi^1)^2-(\xi^2)^2-\Lambda^2\big\} \\ &\qquad +2\xi^3\big\{a_1\xi^1+a_2\xi^2\big\}. \end{aligned} \qquad (3)$$

In the system of equations (2)-(3) the scalar value $s = s(\{\bar{x}\})$ is the *timing parameter*, $\{\bar{x}\} = (x^1,x^2,x^3) \in \mathcal{M}_t$ and $\mathcal{M}_t$ is the tangent bundle of the 3-dimensional Riemannian manifold $\mathcal{M}^{(3)}$, which has a conformally Euclidean metric; $g_{ij}(\{\bar{x}\}) = g(\{\bar{x}\})\delta_{ij}$, where $g(\{\bar{x}\}) = [E-U(\{\bar{x}\})] > 0$, E and $U(\{\bar{x}\})$ are the total energy and total interaction potential of bodies system, respectively, δ_{ij} *is the Kronecker delta function.* The coefficients in (3) are defined by formuals $a_i(\{\bar{x}\}) = -(1/2)\partial_{x^i} \ln g(\{\bar{x}\})$ and $\Lambda(\{\bar{x}\}) = Jg^{-1}(\{\bar{x}\})$, where $J = const$ is the total angular momentum.

Note that the set of coordinates $\{\bar{x}\}$ and the coordinates defined in the Euclidean space $\{\bar{\rho}\} = (\rho_1, \rho_2, \rho_3) \in E^3$ and convenient for determining the interaction potential between bodies are connected by a differential form:

$$\begin{aligned} \rho_1 &= \rho_1^0 + d\rho_1, & d\rho_1 &= \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3, \\ \rho_2 &= \rho_2^0 + d\rho_2, & d\rho_2 &= \beta_1 dx^1 + \beta_2 dx^2 + \beta_3 dx^3, \\ \rho_3 &= \rho_3^0 + d\rho_3, & d\rho_3 &= \gamma_1 dx^1 + \gamma_2 dx^2 + \gamma_3 dx^3, \end{aligned} \quad (4)$$

where $\{\bar{\rho}^0\} = (\rho_1^0, \rho_2^0, \rho_3^0) \in E^3$ denotes the set of coordinates of the initial point. Recall that the sets of coefficients $(\alpha_1, \beta_1, \gamma_1)$ $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ are defined by solution of underdetermined system of algebraic equations:

$$\begin{aligned} \alpha_1^2 + \beta_1^2 + \gamma^{33}\gamma_1^2 &= g(\{\bar{x}\}), \\ \alpha_2^2 + \beta_2^2 + \gamma^{33}\gamma_2^2 &= g(\{\bar{x}\}), \\ \alpha_3^2 + \beta_3^2 + \gamma^{33}\gamma_3^2 &= g(\{\bar{x}\}), \\ \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma^{33}\gamma_1\gamma_2 &= 0, \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma^{33}\gamma_1\gamma_3 &= 0, \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma^{33}\gamma_2\gamma_3 &= 0. \end{aligned} \quad (5)$$

Let note, that this system generates the three-dimensional manifold $\Upsilon^{(3)}$, which, as proved are in homeomorphism with the three-dimensional Euclidean space E^3 and the three-dimensional Riemannian manifold $\mathcal{M}^{(3)}$, respectively.

4. EQUATION FOR PROBABILITY CURRENTS

Let external random forces influence the three-body system. Then the dynamical system, in particular, can be described by stochastic equations of the Langevin type:

$$\dot{\chi}^\mu = A^\mu(\{\chi\}) + \eta^\mu(s), \qquad \mu = \overline{1,6}, \quad (6)$$

where $\{\chi\} = (\{\bar{\xi}\}, \{\bar{x}\})$ and $A^\mu(\{\chi\})$ are regular functions (2), while $\eta^\mu(s)$ - random functions.

Definition 1. The joint probability density for the independent variables can be represented in the form:

$$P(\{\chi\}, s) = \prod_{\mu=1}^{6} \big\langle \delta\big[\chi^\mu(s) - \chi^\mu\big]\big\rangle. \quad (7)$$

where $\delta\big[\chi^\mu(s) - \chi^\mu\big]$ denotes Dirac delta function.

Theorem 1. *If the random functions $\eta^\mu(s)$ satisfy the correlation relations of white noise:*

$$\langle \eta^\mu(s)\rangle = 0, \quad \langle \eta^\mu(s)\eta^\mu(s')\rangle = 2\epsilon\delta(s - s'), \quad \epsilon = const, \quad (8)$$

then, using the equations (6), for the joint probability distribution of geodesic trajectories in the phase space it is possible to obtain the following equation:

$$\frac{\partial P}{\partial s} = \sum_{\mu=1}^{6} \frac{\partial}{\partial \chi^\mu}\Big[A^\mu(\{\chi\}) + \epsilon\frac{\partial}{\partial \chi^\mu}\Big]P. \quad (9)$$

Note that this equation is not an ordinary equation, since, if taking into account the system of algebraic equation (5), the *timing parameter* $s \in \mathcal{M}^{(3)}$ can branching during the evolution of dynamical system, making the equation (9) an irreversible with respect to the *internal time "s"*. The latter generates new geodesic flows with various topological singularities, which further in 6-dimensional phase space can go over into various asymptotic subspaces. In other words, this equation is not a Cauchy problem because of the branching of the timing parameter.

5. A NEW CRITERION FOR THE CHAOS OF THE STATISTICAL SYSTEM

Following the Kullback-Leibler's definition relative to the distance between two continuous distributions, let us define a criterion that indicates on arising deviation between tubes of probabilistic currents of elementary processes [2].

Definition 2. The deviation between of two different tubes of probabilistic currents in the phase space will be defined by the expression:

$$d_{ab} = \int_{\mathcal{P}^6} P(\{\chi\}, s_a) \ln\left|\frac{P(\{\chi\}, s_a)}{P(\{\chi\}, s_b)}\right| \sqrt{g(\{\bar{x}\})} \prod_{\nu=1}^{6} d\chi^\nu, \quad (10)$$

where $P_a \equiv P(\{\chi\}, s_a)$ and $P_b \equiv P(\{\chi\}, s_b)$ two different tubes of probabilistic currents, which at the beginning of development of elementary processes are very close located or have an intersection.

In the case when the distance between two probability flows with time $s = |s_a - s_b|$ grows linearly, that is:

$$d_{ab} = d(s_a, s_b) \sim k|s_a - s_b|, \qquad k = const > 0,$$

then there is every reason to believe that the dynamic system, which is under the influence of the environment, is chaotic. In this case we need to construct mathematical expectation of transition probability between two asymptotic subspaces using the law of large numbers.

Definition 3. If $P_{if}(s_k)$ be the transition probability between the asymptotic channels "i" and "f" with the *internal time* "s_k", then the total mathematical expectation P_{ab}^{tot} will be the sum of partial:

$$P_{if}^{tot} = \lim_{N\to\infty}\Big[\frac{1}{N}\sum_{k=1}^{N}\Big(\lim_{s_k\to\infty} P_{if}(s_k)\Big)\Big]. \quad (11)$$

6. CONCLUSION

The study of the three-body problem, which is a typical example of a dynamical system with all its complexities, has not lost its fundamental significance both for the foundations of physics and for mathematics. In particular, on its basis it is important to answer on principle question, namely: is irreversibility fundamental for describing the classical world [3]? As the study showed, the fundamental irreversibility sits in classical mechanics, starting with three bodies in all non-integrable systems. The latter essentially changes the meaning of the Cauchy problem for numerical simulation of dynamical problems. As proved, to calculate the mathematical expectation of the transition probability between two asymptotic subspaces, it is necessary to combine the solution of the Cauchy problem with the law of large numbers by the formula (11), that is realized by a simple parallel algorithm for computing.

Finally, the developed approach can be a reliable basis for creating high-performance algorithms for mass modeling of multichannel atomic-molecular processes, which will be very valuable for modern technologies.

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