# About One Parallel Algorithm of Solving Non-Local Contact Problem for Parabolic Equations


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## ABSTRACT

In the present work, the initial-boundary problem with non-local contact condition for heat (diffusion) equation is considered. For the stated problem, the existence and uniqueness of the solution is proved. The constructed iteration process allows one to reduce the solution of the initial non-classical problem to the solution of a sequence of classical Cauchy-Dirichlet problems. The convergence of the proposed iterative process is proved; the speed of convergence is estimated. The algorithm is suitable for parallel implementation. The specific problem is considered as an example and solved numerically.



## Keywords

Nonlocal contact problem, iteration method, parallel algorithm, parabolic equation.


**1.** Investigation of nonlocal problems for differential equations of mathematical physics is an important direction of applied and computational mathematics. These problems arise in a natural way during the construction of mathematical models, describing real world processes in different fields of human activity (see [1-11]).

Proposed in this article iterative method for solving stated non-local problem is suitable for parallel implementation and allows us to reduce the solution of non-classical problem to the solution of classical initial-boundary values problems. Note that these problems also can be solved using parallel algorithms (see, for example, [12]-[16]).

**2.** Let us consider the following problem: find the function $u(x,t)$ in the area $\overline{D}=\{(x,t)\,|\,0\le x\le l,\quad 0\le t\le T\}$:

$$u(x,t)=\begin{cases}u^-(x,t), & if\quad 0\le x\le c,\quad 0\le t\le T\\ u^+(x,t), & if\quad c\le x\le l,\quad 0\le t\le T\end{cases}\tag{1}$$

where $0<c<l$, $u^-(c,t)=u^+(c,t),\quad 0\le t\le T$, which satisfies the following conditions:

$$\frac{\partial u^-}{\partial t}=a_1^2\frac{\partial^2u^-}{\partial x^2}-q_1u^-+f^-(x,t),\ \ 0<x<c,0<t\le T, \tag{$2_1$}$$

$$\frac{\partial u^+}{\partial t}=a_2^2\frac{\partial^2u^+}{\partial x^2}-q_2u^-+f^+(x,t),\ \ c<x<l,0<t\le T, \tag{$2_2$}$$

$$\begin{aligned}&u^-(x,0)=u_0^-(x),\quad 0\le x\le c,\\ &u^+(x,0)=u_0^+(x),\quad c\le x\le l,\end{aligned}\tag{3}$$

$$u^-(0,t)=u_1^-(t),\,u^+(l,t)=u_1^+(t),\,0\le t\le T, \tag{4}$$

and the nonlocal contact condition

$$\begin{aligned}u^-(c,t)=u^+(c,t)=u_c(t)=\sum_{i=1}^m\alpha_i^-u^-\left(c_i^-,t\right)+\\ +\sum_{j=1}^n\alpha_j^+u^+\left(c_j^+,t\right)+\varphi_0(t),\quad 0\le t\le T,\end{aligned}\tag{5}$$

where $0<c_m^-<\cdots<c_1^-<c<c_1^+<\cdots c_n^+<l$,

$$\alpha_i^->0,\,i=\overline{1,m},\quad \alpha_j^+>0,\,j=\overline{1,n},\quad \sum_{i=1}^m\alpha_i^-+\sum_{j=1}^n\alpha_j^+\le 0;\tag{6}$$

$f^-(x,t),\ f^+(x,t),\ u_0^-(x),\,u_0^+(x),\ u_1^-(t),\ u_1^+(t),\,\varphi_0(t)$ are known sufficiently smooth functions. $a_1^2,a_2^2$ constants are the coefficients of heat conductivity, $q_1=const>0$, $q_2=const>0$.

Equation (2) describes the distribution of temperature in the homogeneous stem with side heat exchange [17]. These equations are also the equations of diffusion. They describe the process of diffusion in the tube filled with the porous filler, supposing that at any time moment, concentration of gas (solution) in the cross section of a pipe is identical. In this case the coefficients $a_1^2,a_2^2$ are diffusion coefficients [17].

We will call the problem (2)-(5) a nonlocal contact problem for the heat equation.

Our aim is to investigate the problem (2)-(5) and construct an algorithm for its numerical solving.

**3. Theorem 1**. If the regular solution of the problem (2)-(5) exists and the condition (6) is fulfilled, then the solution is unique.

Proof. Suppose that the problem (2)-(5) has two regular solutions: $v(x,t)$ and $w(x,t)$. Then the function $z(x,t)=$ $=v(x,t)-w(x,t)$ is the solution of the following problem:

$$\frac{\partial z^-}{\partial t}=a_1^2\frac{\partial^2z^-}{\partial x^2}-q_1z^-,\ \ 0<x<c,0<t\le T, \tag{$7_1$}$$

$$\frac{\partial z^+}{\partial t}=a_2^2\frac{\partial^2z^+}{\partial x^2}-q_2z^+,\ \ c<x<l,0<t\le T, \tag{$7_2$}$$

$$z^-(x,0)=0,\quad 0\le x\le c,\quad z^+(x,0)=0,\quad c\le x\le l,\tag{8}$$

$$z^-(0,t)=0,\quad z^+(l,t)=0,\quad 0\le t\le T,\tag{9}$$

$$\begin{aligned}z^-(c,t)=z^+(c,t)=z_c(t)=\sum_{i=1}^m\alpha_i^-z^-\left(c_i^-,t\right)+\\ +\sum_{j=1}^n\alpha_j^+z^+\left(c_j^+,t\right),\quad 0\le t\le T.\end{aligned}\tag{10}$$

From the equality (10) it follows that

$$\max_{0\le t\le T}|z_c(t)|\le$$

$$\le \max_{\substack{0\le t\le T\\1\le i\le m}}\left|z^-\left(c_i^-,t\right)\right|\sum_{i=1}^{m}\alpha_i^- + \max_{\substack{0\le t\le T\\1\le j\le n}}\left|z^+\left(c_i^+,t\right)\right|\sum_{j=1}^{n}\alpha_j^+ .$$

Taking into account the condition (6), we obtain

$$\max_{0\le t\le T}|z_c(t)|\le \max_{\substack{0\le t\le T\\1\le i\le m}}\left|z^-\left(c_i^-,t\right)\right| \quad \text{or}$$

$$\max_{0\le t\le T}|z_c(t)|\le \max_{\substack{0\le t\le T\\1\le j\le n}}\left|z^+\left(c_i^+,t\right)\right|.$$

The last inequality means that the function $z^-(x,t)$ does not attain its maximum on the border of the area $\{0\le x\le c,\quad 0\le t\le T\}$ and similarly, the function $z^+(x,t)$ does not attain its maximum on the border of the area $\{c\le x\le l,\quad 0\le t\le T\}$, that contradicts the maximum principle [18]. We can conclude that $z^-(x,t)=0$ in the area $\{0\le x\le c,\quad 0\le t\le T\}$ and $z^+(x,t)=0$ in the area $\{c\le x\le l,\quad 0\le t\le T\}$. So, we obtain that $z(x,t)=0$.

The uniqueness of the solution of the problem (2)-(5) is proved.

**4.** To find a solution of the problem (2)-(5), let us consider the following iteration process:

$$\left[\frac{\partial u^-}{\partial t}\right]^{(k)} = a_1^2\left[\frac{\partial^2 u^-}{\partial x^2}\right]^{(k)} - q_1\left[u^-\right]^{(k)} + f^-(x,t),$$

$$0<x<c,\, 0<t\le T, \qquad (11_1)$$

$$\left[\frac{\partial u^+}{\partial t}\right]^{(k)} = a_2^2\left[\frac{\partial^2 u^+}{\partial x^2}\right]^{(k)} - q_2\left[u^+\right]^{(k)} + f^+(x,t),$$

$$c<x<l,\, 0<t\le T, \qquad (11_2)$$

$$\left[u^-(x,0)\right]^{(k)} = u_0^-(x),\quad 0\le x\le c,$$
$$\left[u^+(x,0)\right]^{(k)} = u_0^+(x),\quad c\le x\le l, \qquad (12)$$

$$\left[u^-(0,t)\right]^{(k)} = \left[u_1^-(t)\right]^{(k)},\ \left[u^+(l,t)\right]^{(k)} = \left[u_1^+(t)\right]^{(k)},$$
$$0\le t\le T, \qquad (13)$$

and the nonlocal contact condition

$$\left[u^-(c,t)\right]^{(k)} = \left[u^+(c,t)\right]^{(k)} = \left[u_c(t)\right]^{(k)} =$$
$$= \sum_{i=1}^{m}\alpha_i^-\left[u^-\left(c_i^-,t\right)\right]^{(k-1)} + \sum_{j=1}^{n}\alpha_j^+\left[u^+\left(c_j^+,t\right)\right]^{(k-1)} + \qquad (14)$$
$$+\varphi_0(t),\quad 0\le t\le T,$$

where $k=0,1,2,\ldots$ and

$$\left[u^-\left(c_i^-,t\right)\right]^{(-1)} = 0,\quad i=\overline{1,m}.,$$
$$\left[u^+\left(c_j^+,t\right)\right]^{(-1)} = 0,\quad j=\overline{1,n}. \qquad (15)$$

**Theorem 2.** If the solution of the problem (2)-(5) exists and the condition (6) is fulfilled, then the iteration process (11)-(15) converges to this solution at the rate of an infinitely decreasing geometric progression.

Proof. Denote

$$\left[z^-(x,t)\right]^{(k)} = \left[u^-(x,t)\right]^{(k)} - u^-(x,t),$$
$$\text{if } \{0\le x\le c,\quad 0\le t\le T\}$$
$$\left[z^+(x,t)\right]^{(k)} = \left[u^+(x,t)\right]^{(k)} - u^+(x,t),$$
$$\text{if } \{c\le x\le l,\quad 0\le t\le T\},$$

where $u^-(x,t)$ and $u^+(x,t)$ are the solutions of the problem (2)-(5).

Then for the function $z^-(x,t)$ and $z^+(x,t)$ we obtain the following problem:

$$\left[\frac{\partial z^-}{\partial t}\right]^{(k)} = a_1^2\left[\frac{\partial^2 z^-}{\partial x^2}\right]^{(k)} - q_1\left[z^-\right]^{(k)},$$
$$0<x<c,\, 0<t\le T, \qquad (16_1)$$

$$\left[\frac{\partial z^+}{\partial t}\right]^{(k)} = a_2^2\left[\frac{\partial^2 z^+}{\partial x^2}\right]^{(k)} - q_2\left[z^+\right]^{(k)},$$
$$c<x<l,\, 0<t\le T, \qquad (16_2)$$

$$\left[z^-(x,0)\right]^{(k)} = 0,\quad 0\le x\le c,$$
$$\left[z^+(x,0)\right]^{(k)} = 0,\quad c\le x\le l, \qquad (17)$$

$$\left[z^-(0,t)\right]^{(k)} = 0,\ \left[z^+(l,t)\right]^{(k)} = 0,\quad 0\le t\le T, \qquad (18)$$

$$\left[z^-(c,t)\right]^{(k)} = \left[z^+(c,t)\right]^{(k)} = \left[z_c(t)\right]^{(k)} =$$
$$= \sum_{i=1}^{m}\alpha_i^-\left[z^-\left(c_i^-,t\right)\right]^{(k-1)} + \sum_{j=1}^{n}\alpha_j^+\left[z^+\left(c_j^+,t\right)\right]^{(k-1)}, \qquad (19)$$
$$0\le t\le T,\quad k=0,1,2,\ldots$$

$$\left[z^-\left(c_i^-,t\right)\right]^{(-1)} = 0,\quad i=\overline{1,m}.,$$
$$\left[z^+\left(c_j^+,t\right)\right]^{(-1)} = 0,\quad j=\overline{1,n}. \qquad (20)$$

If we use Schwarz' Lemma for Heat Equation [19], we can get the following inequalities:

$$\max_{\substack{0\le t\le T\\1\le i\le m}}\left|\left[z^-\left(c_i^-,t\right)\right]^{(k-1)}\right| \le q^- \max_{0\le t\le T}\left|\left[z_c(t)\right]^{(k-1)}\right|,$$

$$\max_{\substack{0\le t\le T\\1\le j\le n}}\left|\left[z^+\left(c_j^+,t\right)\right]^{(k-1)}\right| \le q^+ \max_{0\le t\le T}\left|\left[z_c(t)\right]^{(k-1)}\right|.$$

If we use the nonlocal contact conditions (19), we will have

$$\max_{0\le t\le T}\left|\left[z_c(t)\right]^{(k)}\right| \le Q\max_{0\le t\le T}\left|\left[z_c(t)\right]^{(k-1)}\right|, \qquad (21)$$

where

$$Q = q^-\cdot\sum_{i=1}^{m}\alpha_i^- + q^+\cdot\sum_{j=1}^{n}\alpha_j^+ .$$

Taking into account the conditions (6), we conclude that $0<Q<1$ and

$$\lim_{k\to\infty}\left[z_c(t)\right]^{(k)} = 0 .$$

So, if the solution of the problem (2)-(5) exists, then by the maximum principle [18] we obtain

$$\max_{\substack{0\le x\le c\\0\le t\le T}}\left|\left[u^-(x,t)\right]^{(k)} - u^-(x,t)\right| = O\left(Q^k\right),$$

$$\max_{\substack{c\le x\le l\\0\le t\le T}}\left|\left[u^{+}(x,t)\right]^{(k)}-u^{+}(x,t)\right|=O\left(Q^{k}\right).$$

The theorem is proved.

**Remark.** The iteration algorithm (11)-(15) allows us to reduce the solution of the problem (2)-(5) to the solution of a sequence of Cauchy-Dirichlet problems. This algorithm can be easily implemented on computing system with parallel processors. Note that these classical problems also can be solved using parallel algorithms (see, for example, [12]-[16]).

**5.** Let us prove the existence of a regular solution of the problem (2)-(5) in case $f^{-}(x_1,x_2)\equiv 0$ and $f^{+}(x_1,x_2)\equiv 0$. We will use the iteration process (11)-(15) and introduce the notation $\varepsilon^{(k)}(x,t)=u^{(k)}(x,t)-u^{(k-1)}(x,t)$.

Then the function $\varepsilon^{(k)}(x,t)$ is the solution of the problem (16)-(20) and analogously of the estimation (21) we obtain the following inequality:

$$\max_{0\le t\le T}\left|\left[\varepsilon(c,t)\right]^{(k)}\right|\le Q\max_{0\le t\le T}\left|\left[\varepsilon(c,t)\right]^{(k-1)}\right|,\quad 0<Q<1,$$

or

$$\max_{0\le t\le T}\left|u^{(k)}(c,t)-u^{(k-1)}(c,t)\right|\le$$

$$\le Q\max_{\substack{0\le x\le l\\0\le t\le T}}\left|u^{(k)}(x,t)-u^{(k-1)}(x,t)\right|.$$

So, $u^{(k)}(x,t)-u^{(k-1)}(x,t)\to 0$, when $k\to 0$.

This means that the sequence $\left\{u^{(k)}(x,t)\right\}$ converges uniformly. On the base of Harnak's first theorem [18], the functions $\left[u^{+}(x,t)\right]^{(k)}$ and $\left[u^{-}(x,t)\right]^{(k)}$ converge to the functions $u^{+}(x,t)$ and $u^{-}(x,t)$, which are the solutions of the problem (2)-(6) on the areas $\{0\le x\le c,\quad 0\le t\le T\}$ and $\{c\le x\le l,\quad 0\le t\le T\}$.

We have thereby proved the existence of the regular solution of the problem (2)-(5).

**6.** We consider the following problem as an example: find the function $u(x,t)$ in the area $\{(x,t)\,|\,0\le x\le 1,\ 0\le t\le 3\}$

$$u(x,t)=\begin{cases}u^{-}(x,t), & if\quad 0\le x\le 0.5,\quad 0\le t\le 3\\ u^{+}(x,t), & if\quad 0.5\le x\le 1,\quad 0\le t\le 3\end{cases}$$

which satisfies the equations

$$\frac{\partial u^{-}}{\partial t}=\frac{\partial^2 u^{-}}{\partial x^2}+\left(x^2-x^4\right)\cos t+\left(-3x^4+15x^2-2\right)\sin t,$$

$$0<x<0.5,\quad 0<t\le 3,$$

$$\frac{\partial u^{+}}{\partial t}=\frac{\partial^2 u^{+}}{\partial x^2}-\frac{3}{2}(x-1)x^2\cos t-3\left(x^3-x^2-3x+1\right)\sin t,$$

$$0.5<x<1,\quad 0<t\le 3,$$

the conditions

$$u^{-}(x,0)=0,\quad 0\le x\le 0.5,$$

$$u^{+}(x,0)=0,\quad 0.5\le x\le 1,$$

$$u^{-}(0,t)=0,\quad u^{+}(l,t)=0,\quad 0\le t\le 3,$$

and the nonlocal contact condition

$$u(0.5,t)=0.25u^{-}(0.25,t)+0.25u^{+}(0.75,t)+$$

$$+\frac{123\sin t}{1024},\quad 0\le t\le T.$$

The exact solution of this problem is

$$u(x,t)=\begin{cases}x^2\left(1-x^2\right)\sin t, & 0\le x\le 0.5,\quad 0\le t\le 3,\\ 1.5x^2(1-x)\sin t, & 0.5\le x\le 1,\quad 0\le t\le 3.\end{cases}$$

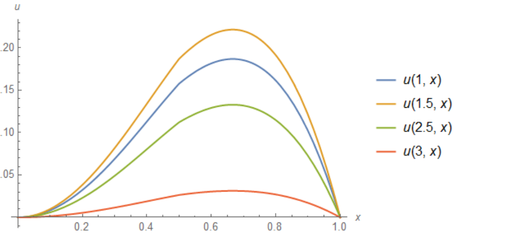


Fig. 1. Exact solution

To solve this problem, we consider the following iteration process:

$$\left[\frac{\partial u^{-}}{\partial t}\right]^{(k)}=\left[\frac{\partial^2 u^{-}}{\partial x^2}\right]^{(k)}+\left(x^2-x^4\right)\cos t+$$

$$+\left(-3x^4+15x^2-2\right)\sin t,\quad 0<x<0.5,\quad 0<t\le 3,$$

$$\left[\frac{\partial u^{+}}{\partial t}\right]^{(k)}=\left[\frac{\partial^2 u^{+}}{\partial x^2}\right]^{(k)}-\frac{3}{2}(x-1)x^2\cos t-$$

$$+3\left(x^3-x^2-3x+1\right)\sin t,\quad 0.5<x<1,\quad 0<t\le 3,$$

$$\left[u^{-}(x,0)\right]^{(k)}=0,\quad 0\le x\le 0.5,$$

$$\left[u^{+}(x,0)\right]^{(k)}=0,\quad 0.5\le x\le 1,$$

$$\left[u^{-}(0,t)\right]^{(k)}=0,\quad \left[u^{+}(l,t)\right]^{(k)}=0,\quad 0\le t\le 3,$$

and the nonlocal contact condition

$$\left[u(0.5,t)\right]^{(k)}=0.25\left[u^{-}(0.25,t)\right]^{(k-1)}+$$

$$+0.25\left[u^{+}(0.75,t)\right]^{(k-1)}+\frac{123\sin t}{1024},\quad 0\le t\le T,$$

where $k=0,1,2,\ldots$ and the initial values for $u^{(k)}(x,t)$ are equal to 0.

Below one can see the figures of approximate solution and respective absolute error for k=1 and k=6.

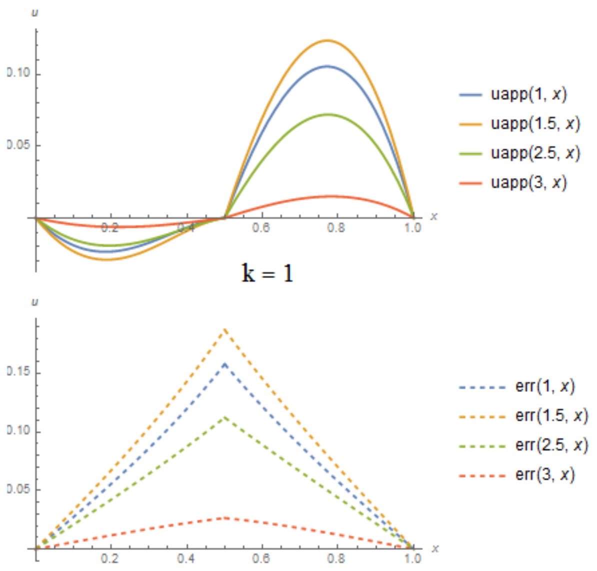


Fig. 2. Approximate solution and abs.error, k=1.

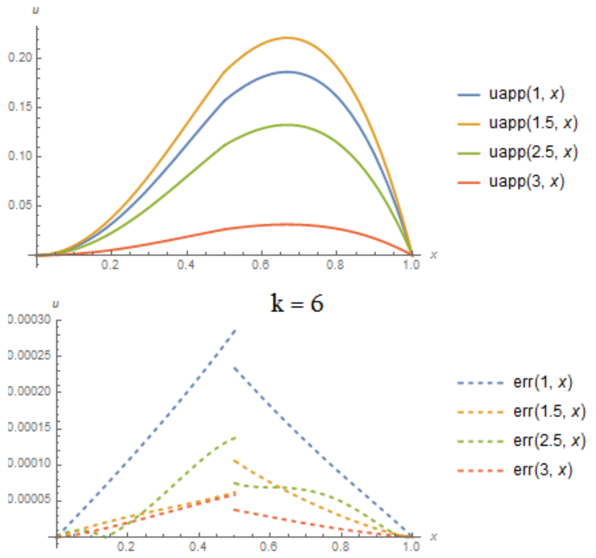


Fig. 3. Approximate solution and abs.error, k=6.

A 8 iterations are enough to find the approximate solution with absolute error <0.0003 for different values of *t*.

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