Digital Solutions of Fredholm First Type Convolution Integral Equations

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ABSTRACT
In this paper a new filtration method, which reduces digital solutions error of Fredholm integral equation of first type is presented.

Keywords
Fredholm, integral equations, solution.

1. INTRODUCTION
When solving many scientific and technological researching problems (thermal conductivity, distribution of potentials), we reduce to different type integral equations solutions.
Here we present the details of integral equations digital computation and the mathematical methods [1, 2].

2. ONE-DIMENSIONAL FREDHOLM EQUATION OF THE FIRST TYPE
Fredholm first type convolution type integral equation represents as

$$\int_{-\infty}^{\infty} k(x-s)y(s)\,ds = f(x), \quad -\infty < x < \infty, \qquad (1)$$

where $k(x) \in L_1(-\infty,\infty)$ is called the kernel of equation, $y(s) \in L_1(-\infty,\infty)$ is unknown function, $f(x) \in L_2(-\infty,\infty)$ is given function.
The classical solution of this equation is obtained by direct and inverse Fourier transforms. Fourier transform is performed on equation (1) left and right sides:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} k(x-s)y(s)\,ds\right]e^{-i\omega x}\,dx =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx. \qquad (2)$$

Denote $k(x), y(s), f(x)$ functions Fourier transforms correspondingly $K(\omega), Y(\omega), F(\omega)$,

$$K(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} k(x)e^{-i\omega s}\,dx,$$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega s}\,dx.$$

Then equation (2) will represent as $K(\omega)\,Y(\omega) = F(\omega)$ or

$$Y(\omega) = \frac{F(\omega)}{K(\omega)}. \qquad (3)$$

Here

$$y(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega s}\,d\omega, -\infty < s < \infty. \qquad (4)$$

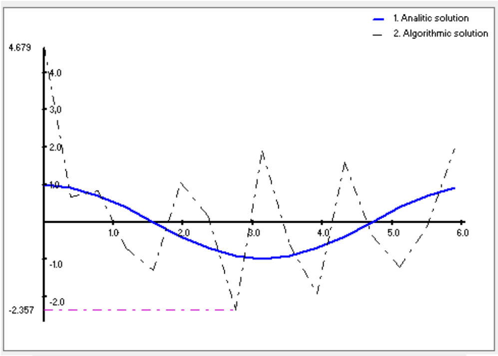
If the following conditions

$$\lim_{\omega\to\infty} K(\omega) = 0, \lim_{\omega\to\infty} F(\omega) = 0, \lim_{\omega\to\infty}\frac{F(\omega)}{K(\omega)} = 0$$

take place and integral (4) converges then the solution of equation (1) exists and it is unique. Since boundaries of integral (4) are unlimited the solution of equation can be obtained only in the case when $F(\omega)$ and $K(\omega)$ can be analytically calculated. Otherwise, it becomes difficult and sometimes impossible to calculate $y(s)$. Considering that instead of function there are often given experimental data, solution obtaining becomes more complicated. The following example illustrates that. Let it is required to obtain the following integral equation digital solutions:

$$\int_{0}^{2\pi} cos(x+t)\,y(t)dt = \pi cos(x).$$

Analitic solution of this equation is function $u(x) = cos(x)$. From formulas (3) and (4) we will obtain the following result.



Max. Error= 3.679250955582

n	x	Analitikal Sol.	Digital Sol.	|Analit.- Digital|
1	0.000000	1.000000000000	4.679250717163	3.679250955582
2	0.392699	0.923879504204	0.666898667812	0.256980895996
3	0.785398	0.707106769085	0.832120180130	0.125013425946
4	1.178097	0.382683426142	-0.686346828938	1.069030284882
5	1.570796	0.000000000000	-1.307646751404	1.307646751404
6	1.963495	-0.382683426142	1.045574069023	1.428257465363
7	2.356194	-0.707106769085	0.143546670675	0.850653469563
8	2.748893	-0.923879504204	-2.356719732285	1.432840108871
9	3.141593	-1.000000000000	1.899477720261	2.899477720261
10	3.534292	-0.923879504204	-0.601772427559	0.322107076645
11	3.926991	-0.707106769085	-1.931837677956	1.224730968475
12	4.319690	-0.382683426142	1.595630049706	1.978313565254
13	4.712389	-0.000000000000	-0.364562571049	0.364562571049
14	5.105088	0.382683426142	-1.219632029533	1.602315425873
15	5.497787	0.707106769085	-0.114041574299	0.821148335934
16	5.890486	0.923879504204	1.990154504776	1.066275000572

Fig. 1 Solid line represents analitic solution, dotted line represents digital solution.

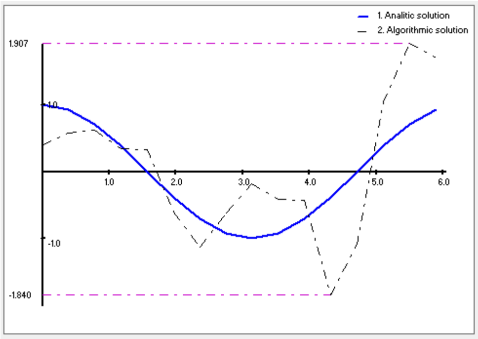
Calculations have been fulfilled for 16-dimensional spectral vectors. Fig. 1 shows the non- correctly problem. Now consider the following form of formula (3):

$$Y(\omega) = \frac{F(\omega)}{K(\omega)+\varepsilon},$$

where $F(\omega)$ *and* $K(\omega)$ are Fourier transforms of functions $\pi cos(x)$ and $cos(x+t)$, $0 < \varepsilon < 1$.
From this spectral representation, it follows that the given equation (1) will represent as

$$\varepsilon y(x) + \int_0^{2\pi} cos(x+t)\, y(t) dt = \pi cos(x).$$

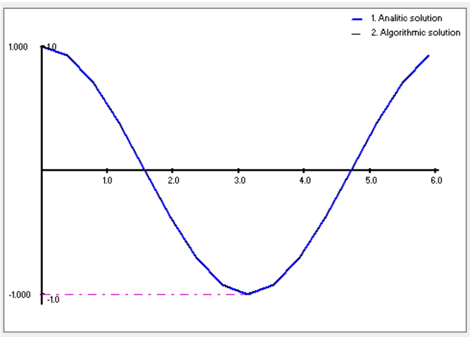
So, Fredholm first type equation reduces to second type equation. If $\varepsilon \to 0$ we will obtain given equation solutions. The best value of ε can be found by applying Tikhonov regularization method to equation [3].
Below an example is given, where $\varepsilon = 10^{-15}$ and $\varepsilon = 10^{-9}$.



Max. Error= 1.457064509392

n	x	Analitikal Sol.	Digital Sol.	|Analit.- Digital|
1	0.000000	1.000000000000	0.405566811562	0.594433188438
2	0.392699	0.923879504204	0.579144895077	0.344734609127
3	0.785398	0.707106769085	0.623216867447	0.083889886737
4	1.178097	0.382683426142	0.347364693880	0.035318743438
5	1.570796	0.000000000000	0.322047173977	0.322047173977
6	1.963495	-0.382683426142	-0.602348208427	0.219664767385
7	2.356194	-0.707106769085	-1.123987555504	0.416880726814
8	2.748893	-0.923879504204	-0.622795522213	0.301084011793
9	3.141593	-1.000000000000	-0.183317899704	0.816682100296
10	3.534292	-0.923879504204	-0.418170899153	0.505708634853
11	3.926991	-0.707106769085	-0.424910932779	0.282195866108
12	4.319690	-0.382683426142	-1.839747905731	1.457064509392
13	4.712389	-0.000000000000	-1.085253357887	1.085253357887
14	5.105088	0.382683426142	1.045144677162	0.662461280823
15	5.497787	0.707106769085	1.906655669212	1.199548840523
16	5.890486	0.923879504204	1.691520333290	0.767640829086

Fig. 2 For $\varepsilon = 10^{-15}$, maximal error is equal to 1.457064509392

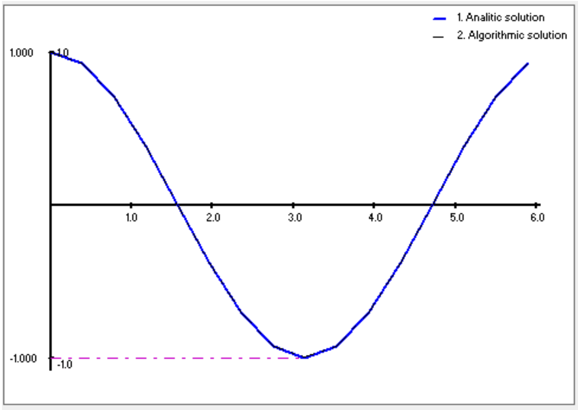


Max. Error= 0.000001290648

n	x	Analitikal Sol.	Digital Sol.	|Analit.- Digital|
1	0.000000	1.000000000000	0.999999582767	4.44e-007
2	0.392699	0.923879504204	0.923879206181	3.26e-007
3	0.785398	0.707106769085	0.707106411457	3.49e-007
4	1.178097	0.382683426142	0.382683157921	2.64e-007
5	1.570796	0.000000000000	0.000000123939	1.24e-007
6	1.963495	-0.382683426142	-0.382683366537	5.25e-008
7	2.356194	-0.707106769085	-0.707106530666	2.25e-007
8	2.748893	-0.923879504204	-0.923878848553	6.98e-007
9	3.141593	-1.000000000000	-0.999999582767	4.44e-007
10	3.534292	-0.923879504204	-0.923879623413	1.19e-007
11	3.926991	-0.707106769085	-0.707106888294	9.49e-008
12	4.319690	-0.382683426142	-0.382684737444	0.000001290648
13	4.712389	-0.000000000000	-0.000000508674	5.09e-007
14	5.105088	0.382683426142	0.382684260607	8.36e-007
15	5.497787	0.707106769085	0.707107901573	0.000001106818
16	5.890486	0.923879504204	0.923880159855	6.34e-007

Fig. 3.For $\varepsilon = 10^{-9}$ the graphs coincide, maximal error is equal to 0.000001290648

To reduce solution error spectrum smoothing is realized in spectral area using ideal filter $sinc(x)$. It is kmown that for 0.5 cyclic frequency filter does not change data being filtering therefore it is realized with frequency $0.5 \pm \varepsilon_0$, $\varepsilon_0 < 0.1$. Kernel's spectrum is smoothing. Below an example of calculation is given.



Max. Error= 2.256656e-007

n	x	Analitikal Sol.	Digital Sol.	|Analit.- Digital|
1	0.000000	1.000000000000	1.000000000000	2.54486174e-008
2	0.392699	0.923879504204	0.923879444599	9.53177661e-008
3	0.785398	0.707106769085	0.707106590271	1.66268876e-007
4	1.178097	0.382683426142	0.382683277130	1.60744193e-007
5	1.570796	0.000000000000	-0.000000049176	4.91760233e-008
6	1.963495	-0.382683426142	-0.382683396339	4.47481519e-008
7	2.356194	-0.707106769085	-0.707106649876	1.20825504e-007
8	2.748893	-0.923879504204	-0.923879384995	1.47301151e-007
9	3.141593	-1.000000000000	-0.999999940395	4.37518572e-008
10	3.534292	-0.923879504204	-0.923879623413	9.13869869e-008
11	3.926991	-0.707106769085	-0.707106888294	1.23691791e-007
12	4.319690	-0.382683426142	-0.382683664560	2.2566563e-007
13	4.712389	-0.000000000000	-0.000000096411	9.64105311e-008
14	5.105088	0.382683426142	0.382683515549	9.75556205e-008
15	5.497787	0.707106769085	0.707106947899	1.69143107e-007
16	5.890486	0.923879504204	0.923879683018	1.28094314e-007

Fig. 4. $\varepsilon_0 = 10^{-7}$, $sinc(x)$ filtration result, maximal error is equal to 2.256656 10^{-7}

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