On the Upper Cone of Degrees Containing Hypersimple *T*-Mitotic Sets Which are not *wtt*-Mitotic

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ABSTRACT

Let us adduce some definitions.
If *A* is a nonrecursive computably enumerable (c.e.) set, then *a splitting of A* is a pair A_1, A_2 of disjoint c.e. sets such that $A_1 \cup A_2 = A$.
A c.e. set A is *T-mitotic* (*wtt-mitotic*) if there is a splitting A_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$ ($A_1 \equiv_{wtt} A_2 \equiv_{wtt} A$).
In this article it is proved, that there exists a low c.e. degree ***u*** such that if ***v*** is a c.e. degree and ***u*** ≤ ***v***, then ***v*** contains a hypersimple *T*-mitotic set, which is not *wtt*-mitotic.

Keywords

Mitotic set, low degree, *T*-reducibility, *wtt*-reducibility, hypersimple set.

1. INTRODUCTION

We'll use notions and terminology introduced in Soare [5].

Notations. We deal with sets and functions over the nonnegative integers $\omega = \{0,1,2,\ldots\}$.

Let φ_e *be* the e^{th} partial recursive function in the standard listing (see Soare [5], p.15, p.25).

If $A \subseteq \omega$ and $e \in \omega$, let $\Phi_e^A(x) = \Phi_e(A:x) = \{e\}^A(x)$ (see Soare [5], pp. 48-50).

χ_A denotes the characteristic function of A, which is often identified with A and written simply as $A(x)$. $f \upharpoonright x$ denotes the restriction of f to arguments $y < x$, and $A \upharpoonright x$ denotes $\chi_A \upharpoonright x$. $\varphi_{e,at\, s+1}(x)\downarrow$ denotes $\varphi_{e,s+1}(x)\downarrow$ & $\varphi_{e,s}(x)\uparrow$. $W_e = dom\,\varphi_e = \{x: \varphi_x(x)\downarrow\}$.

$x \in W_{e,at\, s+1}$ denotes $x \in W_{e,s+1} - W_{e,s}$.

The definitions of pairing function $\tau(x,y) = \langle x, y\rangle$ and *canonical index* y of the given finite set *A* (i.e., $D_y = A$) are given in Soare [5].

Definition 1.2. (*i*) A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a *strong* (*weak*) *array* if there is a recursive function f such that $F_n = D_{f(n)}$ ($F_n = W_{f(n)}$).

(*ii*) An array is *disjoint* if its members are pairwise disjoint.

(*iii*) An infinite set B is *hyperimmune* (*hyperhyperimmune*), if there is no disjoint strong (weak) array $\{F_n\}_{n\in\omega}$ such that $F_n \cap B \neq \varnothing$ for all n.

(*iv*) A c.e. set A is *hypersimple*, abbreviated *h-simple,* (*hyperhypersimple*, abbreviated *hh-simple*) if $\overline{A}$ is hyperimmune (hiperhyperimmune) (see Soare [5], p. 80).

Definition 1.3. If $A = \{a_0 < a_1 < a_2 < \cdots\}$ is an infinite set, the *principal function* of A is $pr(A)$, where $pr(A)(n) = a_n$.

Definition 1.4. (*i*) A degree $\boldsymbol{a} \le \boldsymbol{0}'$ is *low* if $\boldsymbol{a}' = \boldsymbol{0}'$, and *high* if $\boldsymbol{a}' = \boldsymbol{0}''$ (the highest possible value).

(*ii*) A set $A \le_T \varnothing'$ is *low* (*high*), if $\deg(A)$ is low (high) (see Soare [5], p.71).

The definitions of a *dense simple set,* a *strongly hypersimple (sh-simple set),* a *finitely strongly hypersimple (fsh-simple) set* notions are given in Soare [5].

Definition 1.5. A c.e. degree $\boldsymbol{a}$ is *contiguous* if for every pair A, B of c.e. sets in $\boldsymbol{a}$, $A \equiv_{wtt} B$.

Note that each contiguous degree, by definition, doesn't contain *T*-mitotic sets, which are not *wtt*-mitotic.

Ladner and Sasso [3] have proved, that for every nonzero c.e. degree $\boldsymbol{b}$ there is a nonzero c.e. degree $\boldsymbol{a} \le \boldsymbol{b}$ such that $\boldsymbol{a}$ is contiguous.

E. J. Griffiths has proved (see [2]) the following Theorem: *There exists a low c.e. degree* ***u*** *such that if* ***v*** *is a c.e. degree and* ***u*** ≤ ***v***, *then* ***v*** *is not completely mitotic.*

In the theorem below it is proved that there exists an infinite class of degrees, containing hypersimple *T*-mitotic sets, which are not *wtt*-mitotic.

Theorem. *There exists a low c.e. degree* ***u*** *such that if* ***v*** *is a c.e. degree and* ***u*** ≤ ***v***, *then* ***v*** *contains a hypersimple T -mitotic set, which is not wtt-mitotic.*

Notice, that it is impossible to replace the term "*h*-simple" in our theorem with any of the following terms: "dense simple", "*fsh*-simple", "*sh*-simple", "*hh*-simple", "maximal"; since dense simple, *fsh*-simple, *sh*-simple, *hh*-simple, maximal sets are high (Martin [4], see also Soare [5], pp. 210-213).

2. PRELIMINARIES FOR THE THEOREM'S PROOF

Proof. This statement is proved using a finite injury priority argument. We build a set U in stages s, $U = \bigcup_{s\in\omega} U_s$. The set U will be a member of degree ***u***, mentioned in Theorem. We also construct sets $\{V_e\}_{e\in\omega}$ to witness that each c.e. degree in the upper cone of ***u*** contains a T-mitotic but non-*wtt*-mitotic set.

Let γ be a recursive function from ω onto ω^2.

Define (Ψ_i, ψ_i) to be the couple $(\Phi_{i_0}, \varphi_{i_1})$ for all i, where $\gamma(i) = (i_0, i_1)$.

Construct U, $\{V_e\}_{e\in\omega}$ to satisfy, for all $e \in \omega$, the requirements:

N_e : $(\exists^\infty s)(\Phi_e(U;e)[s]\downarrow) \Rightarrow \Phi_e(U;e)\downarrow$.

$R_{\langle e,i\rangle}:(\exists y)(\psi_i(y)\uparrow)$ or $(\exists y)\neg[\Psi_i(V_e\cup\{y\};y)\downarrow=V_e(y)]$ (where Ψ_i is *wtt*-reduction with ψ_i denoting the corresponding use function).

$P_{\langle e,i\rangle}:[[(\forall y)(\psi_i(y)\downarrow)\ \&\ (\forall u,v)((u\neq v)\Rightarrow$

$D_{\psi_i(u)}\cap D_{\psi_i(v)}=\varnothing)]\Rightarrow(\exists y)(D_{\psi_i(y)}\subseteq V_e)]$.

$\tilde{P}_e: W_e=\Lambda^{V_e}$ for some recursive functional Λ.

We also ensure that $V_e\equiv_T U\oplus W_e$ (see the Construction and Lemma 2, Lemma 3).

If $U\leq_T W_e$ then the above condition ensures that $V_e\equiv_T U\oplus W_e\equiv_T W_e$.

If N_e is met for all $e\in\omega$, then U is low.

If $R_{\langle e,i\rangle}$ is met for all $i\in\omega$, then V_e isn't *wtt*-autoreducible. (Note, that a c.e. set A is *wtt*-mitotic if and only if A is *wtt*-autoreducible (see Downey, Stob [2])).

If $P_{\langle e,i\rangle}$ is met for all $i\in\omega$, then V_e is hypersimple.

The proofs of lowness of the set U and the Turing equality of the sets V_e and W_e are similar to the analogous proofs in Theorem 2.2.2 (Griffiths [2]).

The proof of non-*wtt*-mitoticity of V_e uses the proof of nonmitoticity of V_e in Theorem 2.2.2 (Griffiths [2]) with considerable changes.

Definition 2.1. For any set $A\subseteq\omega$ and $x\in\omega$ define the x*-column* of A $A^{(x)}=\{\langle y,z\rangle:\langle y,z\rangle\in A\ \&\ y=x\}$.

Notations. $M_x=\omega^{(x)}$. $M_e^0=M_{2e}$; $M_e^1=M_{2e+1}$;

$M^0=\bigcup_{e=0}^{\infty}M_e^0$; $M^1=\bigcup_{e=0}^{\infty}M_e^1$. Thus, $M^0\cup M^1=\omega$.

We also will ensure that $V_e^0\equiv_T V_e^1$ (where $V_e^0=V_e\cup M^0\ \&\ V_e^1=V_e\cap M^1$).

At each stage s place markers $\lambda(e,x,s)$ on elements of $\overline{V}_{e,s}\cap M_e^0$. Values of λ will be used both as witnesses to prevent possible *wtt*-autoreduction (by the corresponding functionals) for sets V_e and to ensure that W_e is T-reducible to V_e.

Define functions λ,h (at each stage s) in the following way:

Definition 2.2. Initially define $\lambda(e,0,0)=pr(M_e^0)(1)$, $h_0(e,0)=3z_0+2$, $\lambda(e,1,0)=pr(M_e^0)(3(z_0+1)+1)$, where $z_0=\mu y(pr(M_e^0)(3y+2)>pr(M_e^1)(1))$ for all $e\in\omega$.

Let k,z_x be such that $\lambda(e,x,0)=pr(M_e^0)(3k+1)$, $h_0(e,x)=(3z_x+2)$, where $z_x=\mu y(pr(M_e^0)(3y+2)$ $>pr(M_e^1)(3k+1))$, then define

$\lambda(e,x+1,0)=pr(M_e^0)(3(z_x+1)+1)$ for all $e,x\in\omega$.

Also define $h_0(e,x+1)=3z_{x+1}+2$ (as a consequence $\lambda(e,x+2,0)=pr(M_e^0)(3(z_{x+1}+1)+1)$), where $z_{x+1}=\mu y(pr(M_e^0)(3y+2)>pr(M_e^1)(3(z_{x+1}+1)+1)$ for all $e,x\in\omega$.

Also define a function $\xi_s(e,i)$ for all $e,i\in\omega$ (at each stage s). Initially define $\xi_0(e,i)=i$ for all $e,i\in\omega$. We use ξ to ensure that only members of sufficiently large magnitude enter U at stage s, so we can satisfy the lowness requirements N_e.

Order the requirements in the following priority ranking: $N_0,R_0,P_0,N_1,R_1,P_1,\ldots,N_n,R_n,P_n,\ldots$

The $\{\hat{P}_e\}_{e\in\omega}$ do not appear in this ranking.

N_e *requires attention at stage* $s+1$, if it is not *satisfied* and $\Phi_e(U;e)[s]\downarrow$.

$R_{\langle e,i\rangle}$ is *active at stage* $s+1$, if it is not *satisfied* and $(\forall x\leq y)\ (\psi_{i,s}(x)\downarrow)\ \&\ V_{e,s}(pr(M_e^0)(3k))=$ $V_{e,s}(pr(M_e^1)(3k))=0$, where k is such that $y=\lambda(e,\xi_s(e,i),s)=pr(M_e^0)(3k+1)$.

$R_{\langle e,i\rangle}$ *requires attention at stage* $s+1$, if it is not *satisfied* and $V_{e,s}(pr(M_e^0)(3k))=V_{e,s}(pr(M_e^1)(3k))=1$ & $\Psi_i(V_e\cup\{y\};\ y)[s]\downarrow$, where k is such that $y=\lambda(e,\xi_s(e,i),s)=pr(M_e^0)(3k+1)$.

$P_{\langle e,i\rangle}$ *requires attention at stage* $s+1$, if it is not *satisfied* & $(\exists m)[\psi_{i,s}(m)\downarrow\&(\forall z)(z\in D_{\psi_{i,s}(m)}\Rightarrow z\geq pr(M_e^0)h_s(e,i))]$.

We will build $U=\bigcup_s U_s$ and $V_s=\bigcup_s V_{e,s}$ for all $e\in\omega$. Initially all requirements N_e, $R_{\langle e,i\rangle}$ are declared *unsatisfied*.

3. CONSTRUCTION

Stage $s=0$. Let $U_0=\varnothing$, $V_{e,0}=\varnothing$ for all $e\in\omega$.

Stage $s+1$. **Part A.** Act on the highest priority requirement, which requires attention or is active (at stage $s+1$), if such a requirement exists.

Case 1. If N_e requires attention at stage $s+1$, then set $\xi_{s+1}(\hat{e},\hat{i})=\xi_s(\hat{e},\hat{i}+s)$ for each $\langle\hat{e},\hat{i}\rangle\geq e$. This action prevents injury to N_e by lower priority requirements, as we assume that s bounds the use of the halting computation. In this case, of course, $V_{s+1}^*=V_s$, $U_{s+1}=U_s$.

Define $h_{s+1}^*(\hat{e},\hat{i})=h_s(\hat{e},\hat{i}+s)$ for all $\langle\hat{e},\hat{i}\rangle\geq e$.

Declare N_e *satisfied*; declare all lower priority R, N *unsatisfied*.

Case 2 (a). If $R_{\langle e,i\rangle}$ is active at stage $s+1$ via $y=\lambda(e,\xi_s(e,i),s)$, then for the given *e, i* let k_s be such that $y=pr(M_e^0)(3k_s+1)$. Now set

$V_{e,s+1}^*=V_{e,s}\cup\{pr(M_e^0)(3k_s),pr(M_e^1)(3k_s)\}$,

$U_{e,s+1}=U_{e,s}\cup\{pr(M_e^0)(3k_s)\}$.

Define $\lambda^*(e,\xi_s(e,j),s+1)=\lambda(e,\xi_s(e,j+s),s)$ for all $j\geq i+1$. Also define $h_{s+1}(e,j)=h_s(e,j+s)$ for all $j\geq i+1$.

Remark. Let k_s be such that

$\lambda^*(e,\xi_s(e,j),s+1)=pr(M_e^0)(3k_s+1)$ for all $j\geq i+1$. Then $h_{s+1}^*(e,j-1)=pr(M_e^0)(3(k_s-1)+2)$ (see Definition 2.2 and the whole construction).

Thus, if Case 2(a) is applied at stage $s+1$, the marker $\lambda\big(e,\xi_s(e,i),s\big)$ is not moved, but $h_s(e,i)$ is moved.

Declare $R_{\langle e,i\rangle}$ *satisfied*; declare all lower priority R, N *unsatisfied*.

Define the following function $w(e,i,s)$.

Definition 3.1. For the given *e, i* let k_s, n_s be such that

$\lambda(e,\xi_s(e,i),s) = pr(M_e^0)(3k_s+1)$,

$\lambda(e,\xi_s(e,i+s),s) = pr(M_e^0)(3n_s+1)$.

Then define the function $w(e,i,s)$ in the following way:

$w(e,i,0)$ is undefined; $w(e,i,s+1) = pr(M_e^1)(3n_s+1)$, if $pr(M_e^1)(3k_s)$ is included in V_e^1 at stage $s+1$ applying Part A *Case2(a)* and undefined, otherwise.

Remark. Note that if Case 2(a) is applied at stage $s+1$, then $\psi_{i,s}(y)\downarrow$ and $w(e,i,s+1) > \psi_i(y)$ (remind that $y = \lambda(e,\xi_s(e,i),s)$).

Thus, if eventually $\Psi_i\big(V_e \cup \{y\}; y\big)\downarrow$, then the posssible entrance of $w(e,i,s+1)$ in the V_e cannot injure the computation (of $\Psi_i(V_e \cup \{y\}; y)$), because $w(e,i,s+1) > \psi_i(y)$ (if (Ψ_i,ψ_i) indeed would have realized the *wtt*-reducibility).

Case 2 (b). If $R_{\langle e,i\rangle}$ requires attention at stage $s+1$ via

$$y = \lambda\big(e,\xi_s(e,i),s\big) = pr(M_e^0)(3k+1)$$

(it means, that $\Psi_e\big(V_e \cup \{y\};)\,y\big)[s]\downarrow$), then whether $\Psi_e\big(V_e \cup \{y\}; y\big)[s]$ equals 0 or not, we define

$\lambda^*(e,\xi_s(e,\hat{i}),s+1) = \lambda(e,\xi_s(e,\hat{i}+s),s)$ for all $\hat{i} \geq i$.

Define $h^*_{s+1}(e,\hat{i}) = h_s(e,\hat{i}+s)$ for all $\hat{i} \geq i$.

Note, that if we apply Case 2(*b*), then, it means, $(\forall x)\ (x \leq y \Rightarrow \psi_i(x)\downarrow)$.

If $\Psi_e\big(V_e \cup \{y\}; y\big)[s] = 0$, set

$V^*_{e,s+1} = V_{e,s} \cup \big\{y, w(e,i,s_0+1)\big\}$ and $U_{s+1} = U_s \cup \big\{y\big\}$,

where s_0 is such a number, that $s_0 < s$ and at stage s_0+1 the numbers $pr(M_e^0)(3k)$ and $pr(M_e^1)(3k)$ are included in V_e, applying Part A Case 2 (a) for requirement $R_{\langle e,i\rangle}$.

Declare $R_{\langle e,i\rangle}$ *satisfied*; declare all lower priority R, N *unsatisfied*.

Case 3. If $P_{\langle e,i\rangle}$ requires attention at stage $s+1$, then let m_0 be the least of such m, that $\psi_{i,s}(m)\downarrow$ &

$(\forall z)[z \in D_{\psi_{i,s}(m)} \Rightarrow z \geq pr(M_e^0)(h_s(e,i))]$.

If $P_{\langle e,i\rangle}$ is not satisfied, then set

$V^*_{e,s+1} = V_{e,s} \cup D_{\psi_{i,s}(m_0)} \cup \{pr(M_e^0)(h_s(e,i)), pr(M_e^1)(h_s(e,i))\}$

and $U_{s+1} = U_s \cup \{pr(M_e^0)(h_s(e,i))\}$.

Define $\lambda^*(e,\xi_s(e,\hat{i}),s+1) = \lambda(e,\xi_s(e,\hat{i}+s),s)$ for all $\hat{i} \geq i$ and $h^*_{s+1}(e,\hat{i}) = h_s(e,\hat{i}+s)$ for all $\hat{i} \geq i+1$.

Thus, $P_{\langle e,i\rangle}$ is, obviously, satisfied.

Declare all lower priority R, N *unsatisfied*.

Define $\xi_{s+1}(\ ,\ )$, $\lambda^*(\ ,\ ,s+1)$ and $h^*_{s+1}(\ ,)$ not specified in Part A, to be the same as $\xi_s(\ ,\ )$, $\lambda(\ ,\ ,s)$ and $h_s(\ ,)$, respectively.

Part B. Let $x \in W_{s+1} - W_s$, and

$\lambda^*(e,x,s+1) = pr(M_e^0)(3k+1)$ for some *k*.

Then define $V_{e,s+1} = V^*_{e,s+1} \cup \{pr(M_e^0)(3k+1)$, $pr(M_e^1)(3k+1)\}$ and $\lambda(e,x+j,s+1) =$ $\lambda^*(e,\xi_{s+1}(e,x+j+1),s+1)$ for all $j \in \omega$.

Also define $h_{s+1}(e,i+j) = h^*_{s+1}(e,i+j+1)$ for all $j \in \omega$.

Find all $\hat{i}$ such that $\lambda(e,\xi_{s+1}(e,\hat{i}),s+1) \geq \lambda^*(e,x,s+1)$ and declare $R_{\langle e,\hat{i}\rangle}$ *unsatisfied* for each such $\hat{i}$.

Define $\lambda(\ ,\ ,s+1)$, $h_{s+1}(\ ,)$ not specified in Part B above to be the same as $\lambda^*(\ ,\ ,s+1)$, $h^*_{s+1}(\ ,)$, respectively.

4. VERIFICATION

Lemma 4.1. *For all e, i :*

1. N_e *is met and* $\lim_s \xi_s(e,i) = \xi(e,i)$ *exists.*
2. $R_{\langle e,i\rangle}$ *is met and* $\lim_s \lambda(e,\xi_s(e,i),s)$ *exists.*
3. $P_{\langle e,i\rangle}$ *is met and* $\lim_s h_s(e,i) = h(e,i)$ *exists.*

Proof. By induction on $j = \langle e,i\rangle$. Suppose there exists a stage s_0 such that for all $\hat{e},\hat{i} < j$:

1) N_e is met and never acts after stage s_0, $\lim_s \xi_s(\hat{e},\hat{i}) = \xi(\hat{e},i)$ exists and is attained by stage s_0.

2) $R_{\langle\hat{e},\hat{i}\rangle}$ is met and never acts after stage s_0, $\lim_s \lambda(\hat{e},\xi_s(\hat{e},\hat{i}),s)$ exists and is attained by stage s_0.

3) $P_{\langle e,i\rangle}$ is met and never acts after stage s_0, $\lim_s h_s(e,i) = h(e,i)$ exists and is attained by stage s_0.

If 1), 2), 3) take place, then

(1) If N_j ever receives attention after stage s_0, then (applying Part A *Case 1*) it is met and never injured, so there is a stage $s_1 > s_0$, after which its computation does not change from divergent to convergent, and after which N_j does not receive attention. (Else set $s_1 = s_0$.)

$\xi_{s+1}(e,i) = \xi(e,i)$ as $N_0,\dots,N_j$ never again change ξ.

(2) After $s_2 \geq s_1$ when

$W_{e,s_2} \restriction \xi(e,i)+1 = W_e \restriction \xi(e,i)+1$, then $R_{\langle e,i\rangle}$ acts at most twice (probably applying Part A *Case 2*(*a*), Part A *Case 2*(*b*)) and is met, say by stage $s_3 > s_2$. After stage s_2 $R_{\langle e,i\rangle}$ can move $\lambda(e,\xi(e,i),s)$ at most once. Also, for all $\hat{i} \geq i+1$, $R_{\langle e,i\rangle}$ can move $\lambda(e,\xi(e,\hat{i}),s)$ at most twice.

Therefore, $\lambda(e,\xi_{s_3}(e,i),s_3) = \lim_s \lambda(e,\xi_s(e,i),s)$.

(3) If P_j ever receives attention after stage s_3, then it is met and is satisfied forever.

If P_j is satisfied, it doesn't move $h_{s_3}(e,i)$. So $h_{s_3}(e,i)=\lim_s h_s(e,i)=h(e,i)$.

Then $(\psi_i$ is total & $(\forall u)(\forall v)[u\neq v\Rightarrow D_{\psi_i(u)}\bigcap D_{\psi_i(v)}=\varnothing])\Rightarrow$ $(\exists s>s_3)(\exists m)[\psi_{i,s}(m)\downarrow$ & $(\forall z)(z\in D_{\psi_{i,s}(m)}\Rightarrow$ $z\geq pr(M_e^0)h_s(e,i))]$. So, if $(\psi_i$ is total & $(\forall u)(\forall v)[u\neq v\Rightarrow D_{\psi_i(u)}\bigcap D_{\psi_i(v)}=\varnothing])$, then there exists $(s_4>s_3)$ such that $(\exists m)[(\psi_{i,s_4}(m)\downarrow$ & $(\forall z)[z\in D_{\psi_{i,s_4}(m)}\Rightarrow$ $z\geq pr(M_e^0)(h_s(e,i))]$ & $D_{\psi_{i,s_4}(m)}$ is included into V_e at the stage $s_4+1]$. So, $P_{\langle e,i\rangle}$ is met.

If $P_{\langle e,i\rangle}$ is met for all $i\in\omega$, then V_e is hypersimple.

Lemma 4.2. *For all* e, $V_e\leq_T U\oplus W_e$.

Proof. In the construction a number k enters V_e only if a number less than or equal to k enters U or enters W_e, so $V_e\leq_T U\oplus W_e$.

Lemma 4.3. *For all* e, P_e *is satisfied, that is* $W_e=\Lambda^{V_e}$.

Proof. To determine whether $x\in W_e$, we need to find a stage such that $\lambda(e,x,s)$ has attained its limit. Using the oracle V_e, we determine $\lambda(e,0),\ldots,\lambda(e,x)$ (note that $\lambda(e,y,s)$ changes only if a number $\leq\lambda(e,y,s)$ enters V_e). Find a stage s_x such that $V_{e,s_x}\lceil\sigma_x+1=V_e\lceil\sigma_x+1$, where $\sigma_x=\max\{\lambda(e,0),\ldots,\lambda(e,x)\}$. Then $x\in W_e$ iff $x\in W_{e,s_x}$.

Lemma 4.4. V_e *is* T*-mitotic, for all* e.

Proof (sketch). **I**. Prove that $V_e^1\leq_T V_e^0$.

We must determine (using the oracle V_e^0) whether $x\in V_e^1$ or not (for arbitrary number x).

It is obvious, that if $x\in M^0$, then $x\notin V_e^1$.

Let $x\in M^1$. There are the following cases to consider:

(a) If $x\notin M_e^1$, then find z such that $z=\max(\{y\mid x>pr(M_e^1)(y)\})$.

Find a stage s_0 such that $V_{e,s_0}^0\lceil pr(M_e^0)(z)=V_e^0\lceil pr(M_e^0)(z)$. Then $x\in V_e^1\Leftrightarrow x\in V_{e,s_0}^1$. (In this case x can be included in V_e^1 only if $(\exists\, s_1<s_0)(\exists\, i,m)$ such that $x>pr(M_e^0)(h_{s_0}(e,i))$ and x is included in V_e^1 (with $pr(M_e^0)(h_{s_0}(e,i))$, $pr(M_e^1)(h_{s_1}(e,i))$ and $D_{\psi_i(m)}$), applying the Part A *Case 3* at stage s_1+1).

(b) If $x\in M_e^1$, let $x=pr(M_e^1)(k)$ for some k. Find a stage s_0 such that $V_{e,s_0}^0\lceil pr(M_e^0)(k)=V_e^0\lceil pr(M_e^0)(k)$.

Then we'll show
$$x\in V_e^1\Leftrightarrow x\in V_{e,s_0}^1 \qquad (4.5)$$

Note, that for $x=pr(M_e^1)(k)$ (when case *(b)* from Lemma 4.4. I is applied), if x enters into V_e^1 at some stage *t*, then a certain element from $M_e^0\lceil pr(M_e^0)(k)$ enters into set $V_e^0\lceil pr(M_e^0)(k)$ at the same stage *t*.

We'll consider that in details.

Note, that x can be included in V_e^1 only applying Part A *Case 2*(*a*) or Part A *Case* 2(*b*) or Part A *Case 3* or Part B. We'll show, that in all these cases the statement (4.5) takes place.

To prove the statement (4.5) there are three subcases of case *(b)* to consider:

(b$_1$*)* If $(\exists\, s_1<s_0)$ such that x is included in V_e^1 (with some $D_{\psi_i(m)}$) at stage s_1+1, applying Part A *Case 3*, then (4.5) takes place (as in case *(a)* of the proof of Lemma 4.4. I.).

(b$_2$*)* If $(\exists\, s_1<s_0)$ such that x is included in V_e^1 at stage s_1+1, applying Part A *Case 2*(*b*), then $(\exists\, x_1<x)$ (x_1 is included in V_e^0 at stage s_1+1) because of the construction and the definition of the function $w(e,i,s)$ (see Definition 3.1). So (4.5) takes place.

(b$_3$*)* If $(\exists\, s_1<s_0)$ such that the number $x=pr(M_e^1)(k)$ is included in V_e^1 at stage s_1+1, applying Part B or applying Part A *Case 2*(*a*), then the number $pr(M_e^0)(k)$ is included in V_e^0 at stage s_1+1, as it follows from the construction. So (4.5) takes place.

Thus, $V_e^1\leq_T V_e^0$. Therefore, $V_e\leq_T V_e^0$ (because $V_e^0=V_e\bigcap M^0$). But, obviously, $V_e^0\leq_T V_e$ and, therefore, $V_e\equiv_T V_e^0$.

II. Prove that $V_e^0\leq_T V_e^1$.

We must determine (using the oracle V_e^1) whether $x\in V_e^0$ or not (for an arbitrary number x).

It is obvious, that if $x\in M^1$, then $x\notin V_e^0$.

Let $x\in M^0$. There are the following cases to consider:

(a) $x\notin M_e^0$; *(b)* $(\exists k)(x=pr(M_e^0)(3k))$;

(c) $(\exists k)(x=pr(M_e^0)(3k+1)$ & $pr(M_e^1)(3k)\notin V_e^1)$;

(d) $(\exists k)(x=pr(M_e^0)(3k+1)$ & $pr(M_e^1)(3k)\in V_e^1)$.

In all these cases, it is possible to answer the question of whether x belongs to the set V_e^0 according to the methods indicated in the proof of Part I of Lemma 4.4.

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