Invertible Algebras with $\forall\exists^*(\forall)$-Identities of Second Order Logic

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ABSTRACT
In this paper we prove Schauffler's-type theorems for new formulas of the second order logic. This work continues the results of [1, 2, 3], [4] and [5].

Keywords
Quasigroup, $\forall\exists(\forall)$-identity, $\forall\exists^*(\forall)$-identity, universal algebra, isotopy, second order logic.

1. INTRODUCTION
The Invertible algebra is an algebra with quasigroup operations [6, 7]. Let Q be a non-empty set. We denote the system of all binary quasigroup operations and all binary operations defined on the set Q by Ω_Q and G_Q, respectively.

The following formula of the second order logic is called $\forall\exists(\forall)$-identity ([6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]):

$$\forall X_1, \dots, X_k \exists X_{k+1}, \dots, X_m \forall x_1, \dots, x_n (w_1 = w_2), \quad (1)$$

where w_1 and w_2 are the terms, which consist of functional variables $X_1, \dots, X_m$ and object variables $x_1, \dots, x_n$.

In [1] it is proved that in the algebra (Q, Ω_Q) the following $\forall\exists(\forall)$-identity

$$\forall A, B \in \Omega_Q \exists C, D \in \Omega_Q \forall x, y, z \in Q (A(B(x, y), z) = = C(x, D(y, z))) \quad (2)$$

holds if and only if the cardinality of Q is $|Q| \leq 3$.

In [4], this result was modified for groupoids: it was proved that in the algebra (Q, G_Q) the following $\forall\exists(\forall)$-identity

$$\forall A, B \in G_Q \exists C, D \in G_Q \forall x, y, z \in Q (A(B(x, y), z) = = C(x, D(y, z))) \quad (3)$$

holds if and only if the cardinality of Q is $|Q| = 1$ or the set Q is infinite.

In [5], a more general result was proved: in the set Q the following second order formula

$$\forall A, B \in \Omega_Q \exists A', B' \in G_Q \forall x, y, z \in Q (A(B(x, y), z) = = A'(x, B'(y, z))) \quad (4)$$

holds if and only if the cardinality of Q is $|Q| \leq 3$ or the set Q is infinite.

The current paper generalizes the above results and continues the results of [1, 2, 3], [4] and [5]. The following formulas (4), (5),(6),(7),(8) are called $\forall\exists^*(\forall)$-identities. In this paper we study the sets Q, for which the $\forall\exists^*(\forall)$-identities (5),(6),(7),(8) hold in the set Q:

$$\forall A, B \in \Omega_Q \exists A', B' \in G_Q \forall x, y, z \in Q (B'(A(x, y), z) = = A'(x, B(y, z))), \quad (5)$$

$$\forall A, B \in \Omega_Q \exists A', B' \in G_Q \forall x, y, z \in Q (A'(B(x, y), z) = = A(x, B'(y, z))), \quad (6)$$

$$\forall A, B \in \Omega_Q \exists A', B' \in G_Q \forall x, y, z \in Q (A(B'(x, y), z) = = A'(x, B(y, z))), \quad (7)$$

$$\forall A, B \in \Omega_Q \exists A', B' \in G_Q \forall x, y, z \in Q (A(B'(x, y), z) = = B(x, A'(y, z))). \quad (8)$$

We find the necessary and sufficient conditions on the set Q, for which these $\forall\exists^*(\forall)$-identities hold in Q.

2. PRELIMINARY CONCEPTS AND RESULTS
Definition 1. The operations $A, B \in G_Q$ are called isotopic, if there exist permutations α, β, γ of the set Q such that

$$\forall x, y (A(x, y) = \alpha B(\beta x, \gamma y)). \quad (9)$$

Definition 2. *If the operations A and B are isotopic, then algebras $Q(A)$ and $Q(B)$ are called isotopic.*

Obviously the relation of isotopy is an equivalence relation.

In [19] the following lemma is proved:

Lemma 1. *If the loop L and the group G are isotopic, then they are isomorphic.*

In [20] the following fact is stated:

Lemma 2. *If $|Q| \geq 5$, then it is possible to define a loop $Q(L)$, which is not a group.*

In [5] the following lemma is proved:

Lemma 3. *If $|Q| \leq 3$, $A \in \Omega_Q$ and $Q(+)$ is a cyclic group, then $A(x,y) = \alpha x + t + \beta y$, where $t \in Q$ and α, β are automorphisms of the group $Q(+)$.*

3. MAIN RESULTS

Now we can formulate the main results of the current paper.

Proposition 1. *For any non-empty set Q the $\forall\exists^*(\forall)$-identity* (5) *holds.*

Theorem 1. *In the set Q the $\forall\exists^*(\forall)$-identity* (6) *holds if and only if the cardinality of Q is $|Q| \leq 3$.*

Theorem 2. *In the set Q the $\forall\exists^*\forall$-identity* (7) *holds if and only if the cardinality of Q is $|Q| \leq 3$.*

Theorem 3. *In the set Q the $\forall\exists^*\forall$-identity* (8) *holds if and only if the cardinality of Q is $|Q| \leq 3$.*

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