A New Sufficient Condition for a Digraph to be Hamiltonian-A Proof of Manoussakis Conjecture

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ABSTRACT

Y. Manoussakis (J. Graph Theory 16, 1992, 51-59) proposed the following conjecture.

Conjecture. *Let D be a 2-strongly connected digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z, we have $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then D is Hamiltonian.*

In this paper, we confirm this conjecture. Moreover, we prove that if a digraph D satisfies the conditions of this conjecture and has a pair of non-adjacent vertices $\{x, y\}$ such that $d(x) + d(y) \leq 2n - 4$, then D contains cycles of all lengths $3, 4, \dots, n$.

Keywords

Digraph, Hamiltonian cycle, Strong digraph, Pancyclic digraph.

1. INTRODUCTION

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. Every cycle and path are assumed simple and directed; their lengths are the numbers of their arcs. A digraph D is *Hamiltonian* if it contains a cycle passing through all the vertices of D. There are many conditions that guarantee that a digraph is Hamiltonian (see, e.g., [1]-[5]). In [5], the following theorem was proved.

Theorem 1.1: (Manoussakis [5]). *Let D be a strong digraph of order $n \geq 4$. Suppose that D satisfies the following condition for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x)+d(y)+d^+(x)+d^-(z) \geq 3n-2$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2$. Then D is Hamiltonian.*

Definition 1.2: *Let D be a digraph of order n. We say that D satisfies condition (M) when $d(x) + d(y) + d(w)+d(z) \geq 4n-3$ for all distinct pairs of non-adjacent vertices x, y and w, z.*

Manoussakis [5] proposed the following conjecture. This conjecture is an extension of Theorem 1.1

Conjecture 1.3: (Manoussakis [5]). *Let G be a 2-strong digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z we have $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then D is Hamiltonian.*

Manoussakis [5] gave an example, which showed that if this conjecture is true, then the minimum degree condition is sharp. Notice that another examples can be found in [6], where for any two integers $k \geq 2$ and $m \geq 1$, the author constructed a family of k-strong digraphs of order $4k+m$ with minimum degree $4k+m-1$, which are not Hamiltonian. This result disproves a conjecture of Thomassen (see [2], Conjecture 1.4.1. Every 2-strong $(n-1)$-regular digraph of order n, except D_5 and D_7, is Hamiltonian).

Thomassen (see [2]) suggested the following conjectures:

1. (Conjecture 1.6.7 Thomassen [2]): *Every 3-strong digraph of order n and with minimum degree is at least $n + 1$ is strongly Hamiltonian-connected.*

2. (Conjecture 1.6.8 Thomassen [2]): *Let D be a 4-strong digraph of order n such that the sum of the degrees of any pair of non-adjacent vertices is at least $2n + 1$. Then D is strongly Hamiltonian-connected.*

Investigating these conjectures, the author [7] disproved the first conjecture (proving that for every integer $n \geq 9$ there exists a 3-strong non-strongly Hamiltonian-connected digraph of order n with the minimum degree at least $(n + 1)$, and for the second conjecture proved the following theorem.

Theorem 1.4: (Darbinyan [7]). *Let D be a strong digraph of order $n \geq 3$. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of non-adjacent vertices $x, y \in V(D)\backslash\{z\}$, where z is some vertex of $V(D)$. Then D contains a cycle of length at least $n-1$.*

The following corollary immediately follows from Theorem 1.4.

Corollary 1.5: *Let D be a strong digraph of order n satisfying condition (M). Then D contains a cycle of length at least $n-1$ (in particular, D contains a Hamiltonian path).*

In [8], [9] and [10] the authors studied some properties in digraphs with the conditions of Theorem 1.1. The result of [9] gives an answer to a question of Li, Flandrin and Shu [11].

In this paper, we confirm Conjecture 1.3.

Theorem 1.6. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Then D is Hamiltonian.*

We also prove the following theorem.

Theorem 1.7. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that D contains a pair $\{x, y\}$ of non-adjacent vertices such that $d(x) + d(y) \leq 2n - 4$. Then D contains cycles of all lengths $3, 4, \dots, n$.*

Note that Woodall's and Ore's theorems follow from Theorem 1.6.

The proof of Theorem 1.7 is based on Theorem 3.4 and the Moser theorem for a strong tournament to be pancyclic [14].

In view of Theorem 1.7, we set the following problem.

Problem: Let D be a 2-strongly connected digraph of order n satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in D such that $2n-3 \leq d(x) + d(y) \leq 2n-2$. Whether D contains cycles of all lengths $3, 4, \dots, n-1$?

2. TERMINOLOGY AND NOTATION

In this paper we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [1] for terminology and notations not discussed here. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \dots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). Let x and y be two distinct vertices of a digraph D. Cycle that passing through x and y in D, we denote by $C(x, y)$.

A digraph D is *strongly connected* (or just *strong*), if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. A digraph D is k-strongly ($k \geq 1$) connected (or *k-strong*), if $|V(D)| \geq k+1$ and $D\langle V(D) \setminus A\rangle$ is strongly connected for any subset $A \subset V(D)$ of at most $k-1$ vertices.

3. AUXILIARY KNOWN RESULTS

It is not difficult to prove the following lemma.

Lemma 3.1. *Let D be a digraph of order n. Assume that $xy \notin A(D)$ and the vertices x, y in D satisfy the degree condition $d^+(x) + d^-(y) \geq n-2+k$, where $k \geq 1$. Then D contains at least k internally disjoint (x, y)-paths of length two.*

Theorem 3.2 (Meyniel [4]). *Let D be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

Definition 3.3. *For any integers n and m, $(n+1)/2 < m \leq n-1$, let Φ_n^m denote the set of digraphs D, which satisfy the following conditions: (i) $V(D) = \{x_1, x_2, \dots, x_n\}$; (ii) $x_n x_{n-1} \dots x_2 x_1 x_n$ is a Hamiltonian cycle in D; (iii) for each k, $1 \leq k \leq n-m+1$, the vertices x_k and x_{k+m-1} are not adjacent; (iv) $x_j x_i \notin A(D)$ whenever $2 \leq i+1 < j \leq n$ and (v) the sum of degrees for any two distinct non-adjacent vertices is at least $2n-1$.*

Theorem 3.4 (Darbinyan [14]). *Let D be a strong digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n-1$ for all pairs of distinct non-adjacent vertices x, y in D. Then either (a) D is pancyclic or (b) n is even and D is isomorphic to one of $K^*_{n/2,n/2}$, $K^*_{n/2,n/2} \setminus \{e\}$, where e is an arbitrary arc of $K^*_{n/2,n/2}$, or (c) $D \in \Phi_n^m$ (in this case D does not contain a cycle of length m).*

Later on, Theorem 3.4 was also proved by Benhocine [15].

4. PRELIMINARIES

A preliminary version of some results of this section was presented at Emil Artin International Conference [16] and recently published in [17]. We will omit all proofs of lemmas and theorems in this section.

Lemma 4.1: Let D be a digraph of order n satisfying condition (M). Then D contains at most one pair of non-adjacent vertices x, y such that $d(x) + d(y) \leq 2n-2$.

Theorem 4.2: Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(x) + d(y) \leq 2n-2$. Then D is Hamiltonian if and only if D contains a cycle through the vertices x and y.

Theorem 4.3: *Let D be a 2-strong digraph of order $n \geq 3$. Suppose that D contains at most one pair of non-adjacent vertices. Then D is Hamiltonian.*

Lemma 4.4: *Let D be a 2-strong digraph of order $n \geq 3$ and let u, v be two distinct vertices in $V(D)$. If D contains no cycle through u and v, then u, v are not adjacent and there is no path of length two between them. In particular,*

$$d^+(u) + d^-(v) \leq n-2, \quad d^-(u) + d^+(v) \leq n-2$$

$$\text{and} \quad d(u) + d(v) \leq 2n-4.$$

Theorem 4.5: *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(u) + d(v) \leq 2n-2$. Then D is Hamiltonian or D contains a cycle of length $n-1$ passing through u and avoiding v (passing through v and avoiding u).*

Lemma 4.6: Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y) + d(z) \leq 2n-2$ and $C = x_1 x_2 \dots x_{n-k} x_1$ is a cycle in D passing through y and avoiding z, where $2 \leq n-k \leq n-2$. If the subdigraph $D\langle V(D) \setminus V(C)\rangle$ contains a cycle passing through z and $d(y, V(D) \setminus V(C)) = 0$, then D is Hamiltonian.

Lemma 4.7: *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y) + d(z) \leq 2n-2$ and $C = x_1 x_2 \dots x_{n-2} z x_1$ is a cycle of length $n-1$ passing through z and avoiding y in D. Then either D is Hamiltonian or for every $k \in [2, n-3]$, the following holds: $A(\{x_1, \dots, x_{k-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) \neq \emptyset$.*

Lemma 4.8: *Let D be a 2-strong digraph of order $n \geq 3$*

satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y)+d(z) \leq 2n-2$ and $C = x_1x_2 \dots x_{n-2}zx_1$ is a cycle of length $n-1$ passing through z and avoiding y in D. If $x_a \to x_b$ and there are integers l, s, f, t such that $1 \leq l \leq a < s \leq f < b \leq t \leq n-2$ and $\{x_f, x_t\} \to y \to \{x_l, x_s\}$, then D is Hamiltonian.

5. SKETCH OF THE PROOF OF THEOREM 1.6

By Theorem 4.3, the theorem is true if D contains at most one pair of non-adjacent vertices. We may, therefore, assume that D contains at least two distinct pairs of non-adjacent vertices. If the degrees sum of any two non-adjacent vertices is at least $2n-1$, then by Meyniel's theorem, the theorem is true. We may, therefore, assume that D contains a pair of non-adjacent vertices, say y, z, such that $d(y)+d(z) \leq 2n-2$. By Theorem 4.2, to prove the theorem, it suffices to prove that D contains a cycle through y and z. If $d(y)+d(z) \geq 2n-3$, then by Lemma 4.4 we have that D contains a cycle trough y and z, which, in turn, implies that D is Hamiltonian (Theorem 4.2). Thus, we can assume that $d(y)+d(z) \leq 2n-4$. By Theorem 4.5 we have that either D is Hamiltonian or D contains a cycle of length $n-1$ passing through z and avoiding y (passing through y and avoiding z).

Suppose that D is not Hamiltonian, i.e., D contains no cycle through y and z. Let $C := x_1x_2 \dots x_{n-2}zx_1$ be a cycle of length $n-1$ in D, which does not contain y. Then, since D is 2-strongly connected, there are some integers p, q, k, r, $1 \leq p < q \leq k < r \leq n-2$ such that $\{x_k, x_r\} \to y \to \{x_p, x_q\}$ and

$$d(y, \{x_1, \dots, x_{p-1}, x_{q+1}, \dots, x_{k-1}, x_{r+1}, \dots, x_{n-2}\}) = 0;$$

$$d^-(y, \{x_p, \dots, x_{q-1}\}) = d^+(y, \{x_{k+1}, \dots, x_r\}) = 0. \tag{1}$$

Therefore,

$$d(y) = d^+(y, \{x_p, \dots, x_q\}) + d^-(y, \{x_k, \dots, x_r\}) \geq q-p+r-k+2. \tag{2}$$

In order to prove the theorem, it is convenient for D and C to prove the following claims and lemma below (the proofs we omit).

Claim 5.1: *If $p \geq 2$, then $d^-(x_{n-2}, \{z, x_1, \dots, x_{p-1}\}) = 0$.*

Claim 5.2: *Suppose that $k \geq q+1$ and $x_h \to x_l$, where $h \in [q, k-1]$ and $l \in [k+1, n-2]$. Then $d^-(x_k, \{x_1, \dots, x_{q-1}\}) = 0$.*

Claim 5.3: *Suppose that $k \geq q+1$, $x_h \to x_l$ with $h \in [q, k-1]$ and $l \in [k+1, r]$ (possibly, $r = n-2$). Then there is an integer $f \geq 0$ such that $l+f \leq r$, $x_{l+f} \to y$, $d(y, \{x_l, \dots, x_{l+f-1}\}) = 0$ (possibly, $\{x_l, \dots, x_{l+f-1}\} = 0$). Moreover, either there is a vertex x_g with $g \in [l+f+1, n-2]$ such that $x_k \to x_g$ or for any $c \in [h+1, k]$ there is a vertex $x_{c'}$ with $c' \in [c, l-1]$ such that $x_{c'} \to z$.*

Lemma 5.4: *If $p \geq 2$, then $A(\{x_1, \dots, x_{p-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset$.*

Now we are ready to complete the proof of the main result.

By Lemma 5.4, $A(\{x_1, \dots, x_{p-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset$. Similarly, if $r \leq n-3$, then $A(\{x_1, \dots, x_{q-1}\} \to \{x_{r+1}, \dots, x_{n-2}\}) = \emptyset$. Using Lemma 4.8, we obtain $A(\{x_p, \dots, x_{q-1}\} \to \{x_{k+1}, \dots, x_r\}) = \emptyset$. From the last three equalities it follows that

$$A(\{x_1, \dots, x_{q-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset. \tag{3}$$

From (3) and Lemma 4.7 it follows that $k \geq q+1$. Applying Lemma 4.7 on the vertices x_q and x_k, we obtain $A(\{x_1, \dots, x_{q-1}\} \to \{x_{q+1}, \dots, x_{n-2}\}) \neq \emptyset$ and $A(\{x_1, \dots, x_{k-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) \neq \emptyset$. Let $x_a \to x_b$ and $x_h \to x_l$ with $a \in [1, q-1]$, $b \in [q+1, n-2]$, $h \in [1, k-1]$ and $l \in [k+1, n-2]$. Choose b maximal and h minimal with these properties, i.e.,

$$A(\{x_1, \dots, x_{q-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) =$$

$$A(\{x_1, \dots, x_{h-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset. \tag{4}$$

From (3) it follows that $b \leq k$ and $h \geq q$, i.e., $b \in [q+1, k]$ and $h \in [q, k-1]$. If $h \leq b-1$, then $C(y, z) = x_1 \dots x_a x_b \dots x_k y x_q \dots x_h x_l \dots x_{n-2} z x_1$, a contradiction. We may, therefore, assume that $h \geq b$. By Lemma 4.7, $A(\{x_1, \dots, x_{b-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) \neq \emptyset$. Let $x_s \to x_t$, where $s \in [1, b-1]$ and $t \in [b+1, n-2]$. Choose t maximal with this property, i.e.,

$$A(\{x_1, \dots, x_{b-1}\} \to \{x_{t+1}, \dots, x_{n-2}\}) = \emptyset. \tag{5}$$

From (4) it follows that $s \geq q$ and $t \leq k$, i.e., $s \in [q, b-1]$ and $t \in [b+1, k]$. We consider the cases $l \leq r$ and $l \geq r+1$ separately.

Case 1: $l \leq r$.

For this case, it is not difficult to check that the conditions of Claim 5.2 hold. Therefore, there is an integer $f \geq 0$ such that $l+f \leq r$, $x_{l+f} \to y$ and either there is a vertex x_g with $g \in [l+f+1, n-2]$ such that $x_k \to x_g$ or for any $c \in [h+1, k]$ there is a vertex $x_{c'}$ with $c' \in [c, l-1]$ such that $x_{c'} \to z$.

Assume first that $t \geq h+1$. Then, since the arcs yx_p, $x_a x_b$, $x_r y$, $x_h x_l$, $x_k y$, $x_{l+f} y$ are in D and $1 \leq a \leq q-1 < s < b \leq h < t \leq k < l \leq l+f \leq r \leq n-2$, we have that $C(y, z) = x_1 \dots x_a x_b \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{t'} z x_1$, or $C(y, z) = x_1 \dots x_a x_b \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_k x_g \dots x_{n-2} z x_1$ when $x_{t'} \to z$ or when $x_k \to x_g$ respectively. In each case we have a contradiction.

Assume next that $t \leq h$. By Lemma 4.7, $A(\{x_1, \dots, x_{t-1}\} \to \{x_{t+1}, \dots, x_{n-2}\}) \neq \emptyset$. Let $x_{s_1} \to x_{t_1}$, where $s_1 \in [1, t-1]$ and $t_1 \in [t+1, n-2]$. Choose t_1 maximal with this property, i. e.,

$$A(\{x_1, \dots, x_{t-1}\} \to \{x_{t_1+1}, \dots, x_{n-2}\}) = \emptyset. \tag{6}$$

From (5) (respectively, from (4)) it follows that $s_1 \geq b$, i.e., $s_1 \in [b, t-1]$ (respectively, $t_1 \leq k$, i.e., $t \in [t+1, k]$).

If $t_1 \geq h+1$, then $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_k y x_q \dots x_s x_t \dots x_h x_l \dots x_{n-2} z x_1$, a contradiction.

We may, therefore, assume that $t_1 \leq h$. By Lemma 4.7, $A(\{x_1, \dots, x_{t_1-1}\} \to \{x_{t_1+1}, \dots, x_{n-2}\}) \neq \emptyset$. Let $x_{s_2} \to x_{t_2}$, where $s_2 \in [1, t_1-1]$ and $t_2 \in [t_1+1, n-2]$. Choose t_2 maximal with this property, i.e.,

$$A(\{x_1, \dots, x_{t_1-1}\} \to \{x_{t_2+1}, \dots, x_{n-2}\}) = \emptyset.$$

From (6) (respectively, from (4)) it follows that $s_2 \geq t$, i.e., $s_2 \in [t, t_1-1]$ (respectively, $t_2 \leq k$, i.e., $t_2 \in [t_1+1, k]$).

Assume first that $t_2 \geq h+1$. Then it is not difficult to see that $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{t'_2} z x_1$; or $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_k x_g \dots x_{n-2} z x_1$ when $x_{t'_2} \to z$ or when $x_k \to x_g$, respectively. In each case we have a contradiction.

Continuing this process, we finally conclude that for some $m \geq 0$, $t_m \in [h+1,k]$ since all the vertices $x_t, x_{t_1}, \dots, x_{t_m}$ are distinct and in $\{x_{q+1}, \dots, x_k\}$. By the above arguments we have that:

If t_m is odd, then $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_m} x_{t_m} \dots x_k y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{n-2} z x_1$;

If t_m is even, then $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_m} x_{t_m} \dots x_{t'_m} z x_1$ or $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_m} x_{t_m} \dots x_k x_g \dots x_{n-2} z x_1$ when $x_{t'_m} \to z$ or when $x_k \to x_g$, respectively. In all cases we have a cycle passing through y and z, which contradicts our supposition and, hence, the discussion of Case 1 is completed.

Case 2: $l \geq r+1$.

Then $r \leq n-3$. Recall that $h \geq b$ and $x_s \to x_t$, where $s \in [q, b-1]$ and $t \in [b+1, k]$. Note that y, x_h, y, z are two distinct pairs of non-adjacent vertices. We distinguish two subcases: Subcase $t \leq h$ and Subcase $t \geq h+1$. Here, we will consider only the subcase $t \leq h$.

Subcase: $t \leq h$.

Then $b \leq h-1$ since $h \geq t \geq b+1$.

Assume first that $t = h$. Then $x_s \to x_h \to x_l$. By Lemma 4.9, $A(\{x_1, \dots, x_{h-1}\} \to \{x_{h+1}, \dots, x_{n-2}\}) \neq \emptyset$. Let $x_c \to x_d$, where $c \in [1, h-1]$ and $d \in [h+1, n-2]$. From the second equality of (4) it follows that $d \leq k$, i.e., $d \in [h+1, k]$. By (5) we have that $c \geq b$, i.e., $c \in [b, h-1]$. Therefore, $C(y,z) = x_1 \dots x_a x_b \dots x_c x_d \dots x_k y x_q \dots x_s x_h x_l \dots x_{n-2} z x_1$, a contradiction.

Assume next that $t \leq h-1$. Then from the maximality of b and t it follows that $d^-(x_h, \{x_1, \dots, x_{b-1}\}) = 0$. This together with (6) implies that

$$d(x_h) = d^+(x_h, \{x_1, \dots, x_{b-1}\}) + d(x_h, \{x_b, \dots, x_k\}) +$$

$$d^-(x_h, \{x_{k+1}, \dots, x_{l-1}\}) + d(x_h, \{x_l, \dots, x_{n-2}\})$$

$$+ d(x_h, \{z\}) \leq b - 1 + 2k - 2b + l - 1 - k + 2n - 2l$$

$$= 2n - l - 2 + k - b.$$

This together with (2), $d(z) \leq n-1$ and $r \leq n-3$ implies that

$$2d(y) + d(x_h) + d(z) \leq 2q - 2p + 2r - 2k + 4 + 2n - l - 2 + k - b$$

$$+ n - 1 \leq 4n - 2 - (l - r) - (k - q) - (b - q) - 2p,$$

which contradicts condition (M), since $k - q \geq 0$, $b - q \geq 0$. This completes the sketch of the proof of Theorem 1.6. $\square$

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