On the Caps in Affine Space $AG(n, 3)$

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ABSTRACT

A cap in a projective or affine space over a finite field F_q with q elements is a set of points (vectors), no three of which are collinear. We give two new constructions for caps in affine space $AG(n, 3)$, which lead to some new upper and lower bounds on the possible minimal and maximal cardinality of caps in affine space $AG(n, 3)$, respectively.

Keywords

Affine space, cap, points, vectors.

1. INTRODUCTION

In this paper, we consider a variant of the packing problem for the n-dimensional projective geometry $PG(n, q)$ over a finite field F_q with q elements. The packing problem is to find the maximum cardinality of a set points (vectors) with property that no d points from this set are linearly dependent. When d=3 such sets are called caps. The packing problem was first considered by Bose [1], and subsequently Segre [2, 3], obtained comprehensive lower and upper bounds. A cap is called complete when it cannot be extended to a large one. The main problem in the theory of caps is to find the minimal and maximal sizes of complete caps in $PG(n, q)$ or in $AG(n, 3)$, see the survey papers [4, 5, 6] and the references therein. Note that the problem of determining the minimum size of a complete cap in a given space is of particular interest in Coding Theory [5]. If we write the points of the cap as columns of a matrix we obtain a matrix such that every three columns are linearly independent, hence the generator matrix of a linear orthogonal array of strength three. This matrix is a check matrix of a linear code with minimum distance >3. In this paper, we give two new constructions for caps in affine space $AG(n, 3)$.

2. MAIN RESULTS

It is easy to see that if S is a cap in $AG(n, 3)$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \mathbf{0}$ $(mod\ 3)$ for every triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in S$. As in [7, 8], let's denote by $B_n = \{(\alpha_1, \dots, \alpha_n) \,/\alpha_i = 0{,}1\}$ and by P_n the set of points of $AG(n, 3)$ satisfying the following two conditions:

- i) for any triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n$, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \neq \mathbf{0} (mod\ 3)$,
- ii) for any two distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P_n$, there exists i $(1 \leq i \leq n)$ such that $\alpha_i = \beta_i = 2$.

We call P_n to be complete when it cannot be extended to a larger one.

We will define the concatenation of the points in the following way. Let $A \subset AG(n, 3)$ and $B \subset AG(m, 3)$. We form a new set $AB \subset AG(n + m, 3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m})$, where $\boldsymbol{\alpha}' = (\alpha_1, \dots, \alpha_n) \in A$ and $\boldsymbol{\alpha}'' = (\alpha_{n+1}, \dots, \alpha_{n+m}) \in B$. In a similar way one can define the concatenation of the points of three sets, four sets…etc. Note that if $x, y, z \in F_3$, then $x + y + z \equiv 0\ (mod 3)$ if and only if $x = y = z$ or they are pairwise distinct.

It is obvious that $P_1 = \{2\}$ and $P_2 = \{(2, 0), (2, 1)\}$ or $P_2 = \{(0, 2), (1, 2)\}$ and they are complete. Presenting the natural numbers as the sum of three (six) natural numbers and applying Theorem 1 (Theorem 2) [8], one can obtain complete P_n sets for each n.

Theorem 1 [8]**.** The following recurrence relation $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, with initial sets $P_1 = \{2\}$, $P_2 = \{(2, 0), (2, 1)\}$ or $P_2 = \{(0, 2), (1, 2)\}$ and $n = n_1 + n_2 + n_3$, yields complete sets.

Let's form the following ten sets:

$A_1 = P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6}$, $A_2 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6}$
$A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}$, $A_4 = B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}$
$A_5 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_6 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$A_7 = B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_8 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}$, $A_{10} = P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}$.

Theorem 2 [8]. The following recurrence relation $P_n = \bigcup_{i=1}^{10} A_i$, with initial sets $P_1 = \{2\}, P_2 = \{(2, 0), (2, 1)\}$ or $P_2 = \{(0, 2), (1, 2)\}$ and $n = \sum_{i=1}^{6} n_i$, yields complete P_n sets.

Note that the cardinality of P_n, obtained by Theorem 1 (Theorem 2), essentially depends on the representation of n as the sum of three (six) natural numbers. Presenting the natural numbers as the sum of six natural numbers and applying Theorem 2, for some $n \geq 6$ one can obtain larger complete P_n sets than those, which are constructed by Theorem 1.

Using P_n and P_m the second author (Theorems 3, 4) [8] constructed complete caps in $AG(n + m, 3)$, in these paper using P_n we construct caps in the space $AG(n, 3)$.

Let's denote by $O_i = (0, 0, \dots, 0)$ the zero vector with i entries, by $\overline{P_n}$ the set of all inverse points of the points of the set P_n and by $B'_m = B_m \backslash (\overline{P_m} \cap B_m)$.

Theorem 3. If $n = n_1 + n_2 + n_3$ and $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, then $P_n \cup O_{n_1}O_{n_2}B'_{n_3} \cup O_{n_1}B'_{n_2}O_{n_3} \cup B'_m O_{n_2}O_{n_3}$ is a cap in $AG(n, 3)$.

Let's form the following ten sets:

$A'_1 = O_{n_1}O_{n_2}B'_{n_3}B'_{n_4}B'_{n_5}O_{n_6}$, $A'_2 = B'_{n_1}O_{n_2}O_{n_3}O_{n_4}B'_{n_5}B'_{n_6}$
$A'_3 = O_{n_1}B'_{n_2}O_{n_3}B'_{n_4}O_{n_5}B'_{n_6}$, $A'_4 = B'_{n_1}B'_{n_2}O_{n_3}O_{n_4}B'_{n_5}O_{n_6}$
$A'_5 = B'_{n_1}B'_{n_2}O_{n_3}B'_{n_4}O_{n_5}O_{n_6}$, $A'_6 = B'_{n_1}O_{n_2}B'_{n_3}O_{n_4}O_{n_5}B'_{n_6}$
$A'_7 = B'_{n_1}O_{n_2}B'_{n_3}B'_{n_4}O_{n_5}O_{n_6}$, $A'_8 = O_{n_1}B'_{n_2}B'_{n_3}O_{n_4}O_{n_5}B'_{n_6}$
$A'_9 = O_{n_1}B'_{n_2}B'_{n_3}O_{n_4}B'_{n_5}O_{n_6}$, $A'_{10} = O_{n_1}O_{n_2}O_{n_3}B'_{n_4}B'_{n_5}B'_{n_6}$.

Theorem 4. If $n=\sum_{i=1}^{6} n_i$ and $P_n=\cup_{i=1}^{10} A_i$, then $P_n \cup \left(\cup_{i=1}^{10} A'_i\right)$ is a cap in $AG(n,3)$.

Let's denote by B_n^o all points of the set B_n with odd number of 1's entries, by d the minimal number of 2's entries in over all points of the set P_n and by Q'_n the set of all points of the set B_n with less or equal $\lfloor (d-1)/2 \rfloor$ 1's entries.

Theorem 5. If the number of 2's entries in every point of P_n is odd, then $P_n \cup B_n^o$ is a complete cap, otherwise, $P_n \cup Q'_n$ is a cap in $AG(n,3)$.

Theorem 6. For every presentation $n=\sum_{i=1}^{6} n_i$ there are only twelve constructions for P_n and they are presented below.

$P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}$ $P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}$
$P_{n_1}P_{n_2}B_{n_3}P_{n_4}B_{n_5}B_{n_6}$ $P_{n_1}P_{n_2}B_{n_3}P_{n_4}B_{n_5}B_{n_6}$
$P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}$ $P_{n_1}B_{n_2}P_{n_3}B_{n_4}B_{n_5}P_{n_6}$
$P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}$ $P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$P_{n_1}B_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$ $P_{n_1}B_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$
$B_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}P_{n_6}$ $B_{n_1}P_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}$
$B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$ $B_{n_1}P_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}$
$B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$ $B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$
$B_{n_1}B_{n_2}P_{n_3}P_{n_4}P_{n_5}B_{n_6}$ $B_{n_1}B_{n_2}P_{n_3}P_{n_4}P_{n_5}B_{n_6}$
$B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}$ $B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}$

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REFERENCES

[1] R. C. Bose, "Mathematical Theory of the Symmetric Factorial Design", *Sankhya*, Vol. 8, pp. 107-166, 1947.

[2] B. Segre, "Le Geometrie di Galois", *Ann. Mat. Pura. Appl*, Vol. 48, pp. 1-97, 1958.

[3] B. Segre, "Introduction to Galois Geometries", *Ani. Acad. Naz. Lincei Met*., Serie VIII, Vol. VIII, 1967.

[4] J. Bierbrauer, "Large caps", *J. Geome.*, 76, pp. 16-51, 2003.

[5] J. W. P. Hirschfeld and L. Storme, "The packing problem in statistics, coding theory and finite projective spaces", *Proceedings of the fourth Isle of Thorns Conference*, pp. 201-246, 2000.

[6] M. Giulietti, "Small complete caps in Galois affine spaces", *J. Algebr. Comb.*, 25, pp. 149-168, 2007.

[7] K. Karapetyan, "Large Caps in Affine Space $AG(n,3)$", *CSIT Conference 2015*, Yerevan, Armenia, September 28-october 2, pp. 82-83, 2015.

[8] K. Karapetyan, "On the Complete Caps in Galois Affine Space $AG(n,3)$", *CSIT Conference 2017*, Yerevan, Armenia, September 25-29, p. 205, 2017.