Unified Software Platform for Discrete Tomography Reconstruction Algorithms

Ani, Margaryan

Institute for Informatics and
Automation Problems of NAS RA
Yerevan, Armenia
e-mail: ani.margaryan1991@gmail.com

ABSTRACT
In this paper a unified software platform DTR (Discrete Tomography Reconstruction) is introduced created for discrete tomography reconstruction algorithms.

Keywords
Discrete tomography, reconstruction algorithms.

1. INTRODUCTION
Discrete Tomography deals with the recovery of discrete sets from their projections composed along a given set of directions. *Discrete sets* or *lattice sets* are finite subsets of the integer lattice Z^d. The *lattice directions* are represented by any nonzero vectors of Z^d. A line l in d-dimensional Euclidean space is a *lattice line* if it is parallel to a lattice direction and passes through at least one point in Z^d. A *projection* of a lattice set in a lattice direction u is a function giving the number of its points on each line parallel to the direction u. In Discrete Tomography, the typical number of projection directions is two to four ([1]).
Given a set of lattice directions $\{u_1, u_2, \cdots, u_l\}$ and projections along those directions: $F_1, F_2, \dots, F_l$. Consider *Consistency* and *Reconstruction* problems in Discrete Tomography.
Consistency: Does there exist a discrete set $T \in Z^d$ with given projections $F_1, F_2, \dots, F_l$ in lattice directions $u_1, u_2, \cdots, u_l$?
Reconstruction: Construct a discrete set $T \in Z^d$ from its projections $F_1, F_2, \dots, F_l$.
These are NP-hard problems for $d \geq 2$ and $l \geq 3$ non-parallel projections in the integer lattice Z^d ([2]).
Due to the complexity of the problem, a special attention has been given to the 2-dimensional case. Subsets of Z^2 can be presented as binary images or binary matrices, where the 1s determine the cells of T. Various studies are devoted to the case of orthogonal projections: horizontal and vertical. In terms of binary matrices, the row sum corresponds to the horizontal projection of T, and the column sum corresponds to the vertical projection. With only horizontal and vertical projections the problem has polynomial complexity ([3]), but the number of solutions can be large ([4]). Any prior knowledge /constraint/ about the image to be reconstructed, can reduce the search space of possible solutions. Using geometrical knowledge about discrete sets, such as convexity and connectedness, is a well-studied area ([2], [5], [6]). In most cases these are NP-complete problems, in the meantime, the existence/construction problems for horizontally and vertically convex and connected matrices can be solved in polynomial time ([7]).
Another property of discrete sets, coming from their binary matrix representation is the non-repetitiveness of the matrix rows [8], which appears also in a number of applications (such as the design of experiments [9]). But mainly, this property/constraint leads to hard or open problems in terms of complexity.
A number of studies are devoted to the case of orthogonal and diagonal projections [2], [10]. In general, the problem of existence/reconstructing of binary images from the given orthogonal and diagonal projections is NP-complete [2]. Also here, the case of horizontal-vertical-diagonal connected and convex sets has polynomial complexity [10].
In this paper, we introduce a unified software platform DTR (Discrete Tomography Reconstruction) created for discrete tomography reconstruction algorithms, where a set of existing representative algorithms are implemented. DTR is developed using modern programing languages (JAVA, JS, HTML, CSS, REST API) and has cross platform support (UNIX, WINDOWS, OSX), which is very flexible for adding new algorithms implementations into the platform. Another advantage is its reusability in other systems.
The rest of the paper is organized as follows: In Section 2 below a brief summary of reconstruction problems for different sets of projections, as well as short descriptions of corresponding algorithms, which have been implemented in DTR are given. Section 3 describes the platform in general and some implementation details.

2. PROBLEM DEFINITION AND RECONSTRUCTION ALGORITHMS

2.1. Orthogonal projections
Consider T, a finite set in the two-dimensional integer grid Z^2, and let A be its binary matrix representation, where 1s in the matrix determine the cells of T. $R = (r_1, \cdots, r_m)$ and $S = (s_1 \cdots, s_n)$ are the row and column sums of A, where $r_i = \sum_{j=1}^{n} a_{i,j}$, $i = 1, \cdots, m$ and $s_j = \sum_{i=1}^{m} a_{i,j}$, $j = 1, \cdots, n$.
Obviously R corresponds to the horizontal projection of T, and S corresponds to the vertical projection.
Consistency: Given positive integer vectors $R = (r_1, \cdots, r_m)$ and $S = (s_1 \cdots, s_n)$. Does there exist a binary matrix of size $m \times n$ with row sum R and column sum S?
Reconstruction: Construct a binary matrix from its row and column sum vectors.
The consistency and reconstruction problems are solved by Ryser in [3], where also a reconstructing algorithm of complexity $O(mn)$ is provided. The outline of the Ryser's algorithm is as follows. Given row and column sum vectors $R = (r_1, \cdots, r_m)$ and $S = (s_1 \cdots, s_n)$. Firstly the "maximal matrix" $\bar{A}$ is constructed, where each of its m rows has the following structure: $\overbrace{1,1,\cdots,1}^{r_i}\overbrace{0,0,\cdots,0}^{n-r_i}$ (r_i 1s followed by $n - r_i$ 0s) for $1 \leq i \leq m$. Then, the necessary number of 1s in columns (s_i 1s in the i-th column) of the required matrix is provided step by step (column by column, starting from the last column) by moving 1s to the current column (within the same row).

The first algorithm, which is implemented in the unified software platform DTR, is the Ryser's algorithm, as a classic algorithm in this domain.

2.2. Orthogonal projections with geometrical constraints

Recall that with only horizontal and vertical projections the number of possible solutions can be exponentially large ([4]). Convexity and connectedness are two commonly used geometrical properties of discrete sets /binary images/ to narrow the class of possible solutions.

A binary matrix is h*-convex*, if the 1s in every row form an interval; and it is v*-convex* if the 1s in every column form an interval. A binary matrix is hv*-convex* if it is both h-convex and v-convex. A binary matrix is *connected*, if the 1s are connected with respect to the adjacency relation (4-adjacency, where vertical and horizontal neighbors are taken into account, and 8-adjacency, where vertical, horizontal and diagonal neighbors are taken into account).

In most cases the consistency and reconstruction of binary matrices with given orthogonal projections and with convexity/connectivity properties are NP-complete problems, however, the existence/construction problems for hv-convex and connected matrices can be solved in polynomial time; an algorithm is introduced in [7] for reconstructing of hv-convex connected matrices (with respect to 4-adjacency). The algorithm first constructs a 2-Satisfiability (2-SAT) Boolean expression such that it is satisfiable if and only if there exists a horizontally and vertically convex and connected matrix with given orthogonal projections. In the final step the algorithm only needs to solve the 2SAT expression. A similar algorithm is introduced in [11] for the case of 8-adjacency. These are the next two algorithms implemented in DTR.

2.2. Orthogonal and diagonal projections

Consider a binary matrix $A = \{a_{i,j}\}$ with m rows and n columns. Let $D = (d_1, \dots, d_{m+n-1})$ denote the diagonal sum vector of A, where:

$d_k = \sum_{i+j=k+1} a_{ij}\,,\ k = 1, \dots, m+n-1.$

The anti-diagonal sum vector $D^A = (d_1^A, \cdots, d_{m+n-1}^A)$ can be defined accordingly.

In case of horizontal, diagonal and vertical projections the reconstruction problems of hvd-connected (with respect to 8-adjacency) and convex (in the horizontal, vertical and diagonal directions) matrices can be solved in polynomial time; the algorithm uses polynomial transformation of the reconstruction problem to the 2-SAT problem ([10]).

The next algorithm, which is implemented in DTR, is the algorithm for reconstructing binary images from given horizontal and diagonal projections. Complexity of the problem is not known, and we consider here a polynomial time heuristic algorithm HD developed in [12].

Similar to Ryser's algorithm, it first constructs the maximal matrix $\bar{A}$, and then the necessary number of 1s in diagonals of the required matrix is provided step by step (diagonal by diagonal) by moving 1s to the current diagonal (within the same row) to get a required matrix.

3. PLATFORM FEATURES AND EXPERIMENTAL DATA

In this section, we introduce version 1.0 of the unified software platform DTR, which is designed for solving discrete tomography reconstruction problems having as an input the projections of a binary matrix/image. In this version two main parts are distinguished: one of them is considered to work with orthogonal projections; and the other - to work with orthogonal and diagonal projections. Unified user interface is given in Figure 1 below.

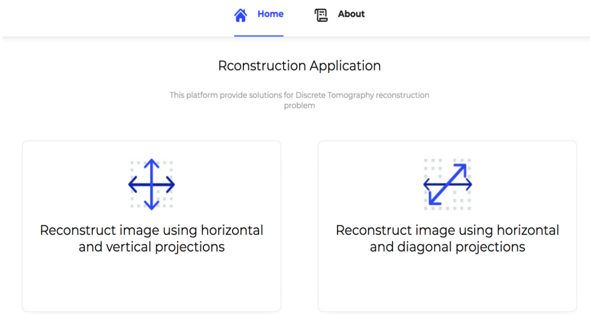

Figure 1
DTR user interface

DTR is developed using modern programing languages (JAVA, JS, HTML, CSS, REST API) and has cross platform support (UNIX, WINDOWS, OSX).

Now we will go through the description of DTR features.

Input is the possibility to input projections of an image under reconstruction, and also - geometrical properties of the image.

Compatibility checking provides possibility to check the compatibility of inputted projections. In case of orthogonal projections the compatibility of given pair of projection vectors assumes general maximum values and sums equality check:

$$\sum_{k=1}^{n} s_k = \sum_{i=1}^{m} r_i$$
$$0 \le r_i \le n,\ \ 1 \le i \le m$$
$$0 \le s_k \le m\,, 1 \le k \le n$$

Besides the simple compatibility checking of input vectors, there is a need to check also the "majorization condition" which is a necessary and sufficient condition for existence of a matrix with given orthogonal projections ([3]). This is provided before the algorithm starts to work.

In the case when also diagonal projections are inputted, the compatibility checking of input vectors assumes the following:

$$\sum_{k=1}^{m+n-1} d_k = \sum_{i=1}^{m} r_i$$
$$0 \le r_i \le n,\ \ 1 \le i \le m$$
$$0 \le d_k \le m_k\,, 1 \le k \le m+n-1,$$

where

$$m_k = \begin{cases} k, & if\ \ 1 \le k \le \min(m,n) \\ \min(m,n), & if\ \ \min(m,n)+1 \le k \le \max(m,n)-1 \\ \min(m,n) - (k - \max(m,n)), & if\ \ \max(m,n) \le k \le m+n-1 \end{cases}$$

Additionally, we need to check also the "majorization conditions", which are necessary conditions for the existence of a matrix with given horizontal and diagonal projections [12]. Those conditions are related to different fragments of the maximal matrix $\bar{A}$, and it is worth mentioning that they are to be checked also in every step, because if one of them is violated, then the required matrix cannot be reconstructed.

Visualization is a common unified visualization component, based on SVG (Scalable Vector Graphics) technology, which is an image representation technology with vectorization implementation. It enables to visualize the final constructed matrix, as well as intermediate matrices of current steps if necessary.

Additionally, Debugging provides possibility to follow the moves in each step, which is helpful in the algorithms development stage.

Testing/experiments is mainly designed for the algorithm HD to assess its performance although it can be applied to all the algorithms of the platform.

The following cases are conducted:

- Input is a pair of random vectors.

In this case random vectors are generated, and then compatibility of the vectors, as well as the necessary conditions are checked. For keeping randomness there is an option to insert the matrix size and rate of each component of the row and diagonal sum vectors comparative to its maximal value.

- Input is row and diagonal sum vectors of random binary matrices.

For this purpose random matrices are generated and then row and diagonal sums are calculated. To keep randomness in generating process an option is created to insert matrix size and probability of each matrix cell (to be 1).

- Input is row and diagonal sum vectors inserted manually.

The purpose here is to check the algorithm performance for specially created test cases of row and diagonal sums.

Consider examples.

An implementation of Ryser's algorithm for the given projection vectors $H = (5,4,3,2,2)$ and $V = (4,4,3,3,2)$ is given in Figure 2; the image in the left part corresponds to the maximal matrix, and the image in the right part corresponds to the finally constructed matrix.

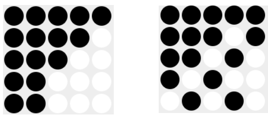

Figure 2
Implementation of Ryser's algorithm for
$H = (5,4,3,2,2), \quad V = (4,4,3,3,2)$

The next example given in Figure 3, corresponds to the construction of hv-convex 4-connected image (solving 2SAT expression).

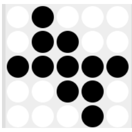

Figure 3
hv-convex 4-connected image for
$H = (1,2,5,2,1)$, $V = (1,3,3,3,1)$

The second part of the platform works with the orthogonal and diagonal projections. Figure 4 demonstrates the implementation of the algorithm HD, where the input row and diagonal sum vectors are taken from a generated random binary matrix:

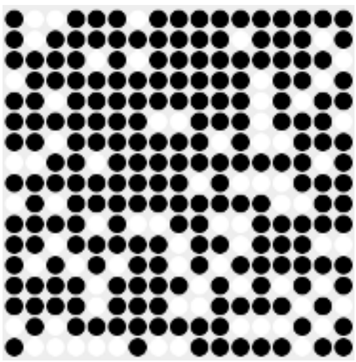

Figure 4: (a)

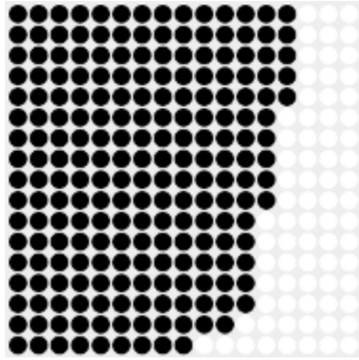

Figure 4: (b)

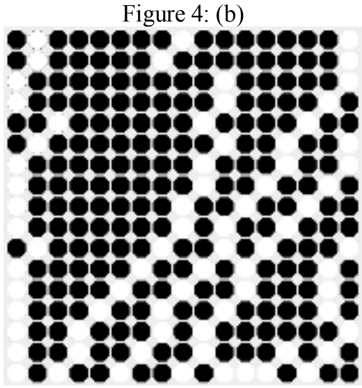

Figure 4: (c)

Figure 4. Stages of performance of algorithm HD: (a) generated random matrix, (b) created maximal matrix, (c) reconstructed image

Figure 5 demonstrates the performance of algorithm HD.

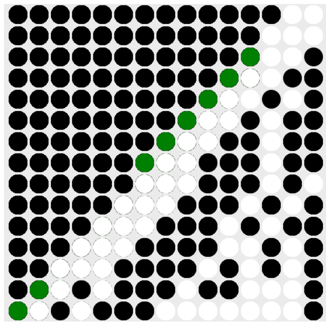

Figure 5: (a)

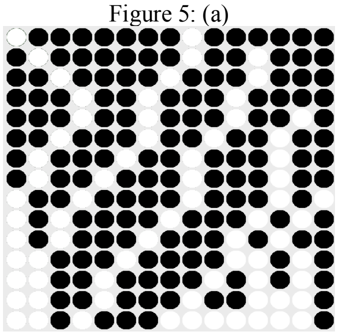

Figure 5: (b)

Figure 5
Performance of algorithm HD with debugging:
(a) current step, where only a part of an image is reconstructed; cells in green show the 1s to be moved to the current diagonal. (b) final reconstructed image.

4. ACKNOWLEDGEMENT

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5. CONCLUSION

In this paper, a unified software platform DTR (Discrete Tomography Reconstruction) is introduced created for discrete tomography reconstruction algorithms, where some existing representative algorithms are implemented. As a future work more algorithms implementation is planned to be added to the platform. Also, more testing and experimentation will be added including testing on real projections data collected from different spheres of discrete tomography use cases.

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