Some Non-linear Version of a Non-local Problem and Its Discrete Analogy

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ABSTRACT

In this communication a non-local modified characteristic problem for a second order quasi-linear equation with real characteristics is investigated.

Keywords

Non-local problem, hyperbolic equation, difference scheme.

We investigate a modified characteristic problem for a second order quasi-linear equation with real characteristics. As it is known, characteristics of quasi-linear equations may depend on values of an unknown solution and its derivatives. In such a case they are unknown and should be defined simultaneously with a solution (see [1]). Characteristics of such equations may form families of any geometry. However, there exist equations, for which these characteristic families may have a quite definite configuration.

Let us consider one non-local problem and its discrete analogue for non-strictly hyperbolic equation

$$u_{xx} + (1 + u_x + u_y)u_{xy} + (u_x + u_y)u_{yy} = 0. \qquad (1)$$

The characteristic roots of (1) are $\lambda_1 = 1$ and $\lambda_2 = u_x + u_y$. (1) is a hyperbolic equation, but in the case $u_x + u_y = 1$ it degenerates parabolically. Therefore, the class of hyperbolic solutions of the considered equation should be defined by the condition $u_x + u_y - 1 \neq 0$.

If we know the value of the sum

$$u_x(x_0, y_0) + u_y(x_0, y_0) \equiv \alpha(x_0, y_0)$$

for some set of points (x_0, y_0) from the plane $\mathbf{R}^2$, then the characteristics of the family of the root λ_2 are representable as $y - y_0 = \alpha(x_0, y_0)(x - x_0)$.

If $\alpha(x_0, y_0) = 1$, then the equation parabolically degenerates all over the straight line $y - y_0 = x - x_0$. If the condition

$$\alpha(x, 0) \neq 1,\ x \in [0, a] \qquad (2)$$

is fulfilled, the characteristics of the family of the root λ_2 have the form $y = \alpha(x_0, 0)(x - x_0)$, $x_0 \in [0, a]$, and they intersect with the straight line $y = x$ at the point $(\mu(x_0), \mu(x_0))$, where

$$\mu(x) = \frac{x\alpha(x, 0)}{\alpha(x, 0) - 1}.$$

The Darboux type nonlocal problem. *Find a regular solution $u(x, y)$ of equation (1) and, along with it, the domain of its propagation if it satisfies condition (2) and the nonlocal condition*

$$u(x, 0) + \beta(x)\, u(\mu(x), \mu(x)) = \varphi(x), \quad x \in [0, a], \quad (3)$$

where $\alpha, \beta, \varphi \in C^2[0, a]$ are given functions and $\alpha(x) \equiv \alpha(x, 0)$.

The similar problems were studied by different authors (see, for example, [2]). We formulate without proof theorems, in which the solution of the problem (1,3) is investigated.

Theorem 1. Let the following conditions be fulfilled:

$$|2\alpha(x) - 1| > 1, -\infty < \alpha'(x) < 0, x \in [0, a], \beta(0) \neq 1. \quad (4)$$

Then there exists a unique regular solution of the nonlocal problem (1,3) in a characteristic triangle bounded by the data supports and characteristic $y = \alpha(a)(x - a)$.

Theorem 2. Let the conditions (4) be fulfilled and the system

$$x = \frac{z - t\alpha(t)}{1 - \alpha(t)}, \quad y = \frac{(z - t)\alpha(t)}{1 - \alpha(t)}$$

define a unique inverse transformation from the arguments z, t to the arguments x, y:

$$z = y - x,\ t = R(x, y).$$

Then in the characteristic triangle there exists a unique regular solution of the nonlocal problem (1–3), which is given by the formula

$$u(x, y) = \int_0^x (\alpha(\tau) - g(\tau))\, d\tau +$$

$$+ \int_0^y \left(g(x - \tau) + \log \frac{\alpha(x - \tau) - 1}{\alpha(R(x, \tau)) - 1} \right) d\tau,$$

where

$$g(x) = \alpha(x) - \varphi'(x) -$$

$$-\beta'(x) \left(u(0, 0) + \int_0^x \alpha(t)\mu'(t)\, dt \right) -$$

$$-\beta(x)\alpha(x)\mu'(x). \qquad (5)$$

Theorem 3. The problem (1–3) is equivalent to the following initial problem: let us find the regular solution of (1) if the following conditions are satisfied:

$$u(x,0) = f(x), \quad u_y(x,0) = g(x), \tag{6}$$

where

$$f(x) = \varphi(x) + \beta(x) \cdot \left(u(0,0) + \int_0^x \alpha(t) \cdot \mu'(t)\,dt \right)$$

and g is given by (5).

The proofs of these theorems can be found in [3].

Below is given the graph of the exact solution of problem (1-3) for specific values of functions α, β and φ such that the conditions of Theorem 1 are satisfied.

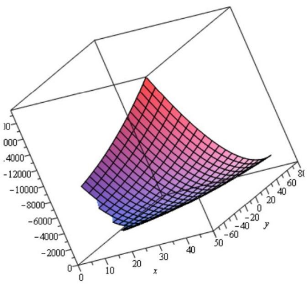

Figure 1: The solution of the problem (1-3), $\alpha = -2x - 2$, $\beta = 2$, $\varphi = x$, $a = 50$.

Let us consider the discrete analog of the problem (1–3). According to Theorem 3, we claim that the problem (1–3) is equivalent to the problem (1),(6). Let us formulate a discrete analog of the problem (1), (6), which is based on the numerical characteristic method.

Let us define a regular mesh on the interval $[0,a]$:

$$\omega_h = \{\widetilde{x}_i, \quad \widetilde{x}_i = ih, \quad i = 0,1,\ldots,N, \quad Nh = a\}$$

The following system is a difference analog of the characteristic differential forms of (1):

$$\widetilde{y}_i^{(n+1)} - \widetilde{y}_i^{(n)} = \widetilde{x}_i^{(n+1)} - \widetilde{x}_i^{(n)},$$

$$\widetilde{y}_i^{(n+1)} - \widetilde{y}_{i+1}^{(n)} = (\widetilde{p}_i^{(n)} + \widetilde{q}_i^{(n)})(\widetilde{x}_i^{(n+1)} - \widetilde{x}_{i+1}^{(n)}),$$

$$\widetilde{p}_i^{(n+1)} - \widetilde{p}_i^{(n)} = (\widetilde{p}_i^{(n)} + \widetilde{q}_i^{(n)})(\widetilde{q}_i^{(n+1)} - \widetilde{q}_i^{(n)}),$$

$$\widetilde{p}_i^{(n+1)} - \widetilde{p}_{i+1}^{(n)} + \widetilde{q}_i^{(n+1)} - \widetilde{q}_{i+1}^{(n)} = 0,$$

$$\widetilde{u}_i^{(n+1)} = \frac{1}{2}\widetilde{p}_i^{(n)}(\widetilde{x}_i^{(n+1)} - \widetilde{x}_i^{(n)}) +$$

$$\frac{1}{2}\widetilde{q}_i^{(n)}(\widetilde{y}_i^{(n+1)} - \widetilde{y}_i^{(n)}) +$$

$$+\frac{1}{2}(\widetilde{p}_{i+1}^{(n)}(\widetilde{x}_i^{(n+1)} - \widetilde{x}_{i+1}^{(n)}) + \widetilde{q}_{i+1}^{(n)}(\widetilde{y}_i^{(n+1)} - \widetilde{y}_{i+1}^{(n)})),$$

$$\widetilde{x}_i^{(0)} = \widetilde{x}_i, \quad \widetilde{y}_i^{(0)} = 0, \quad \widetilde{u}_i^{(0)} = u(\widetilde{x}_i, 0),$$

$$\widetilde{p}_i^{(0)} = f'(\widetilde{x}_i, 0), \quad \widetilde{q}_i^{(0)} = g(\widetilde{x}_i, 0),$$

$$i, n = 0,1,\ldots,N-1, \quad Nh = a.$$

We realize the recalculation by the following scheme:

$$\widetilde{y}_i^{(n+1)} - \widetilde{y}_i^{(n)} = \widetilde{x}_i^{(n+1)} - \widetilde{x}_i^{(n)},$$

$$\widetilde{y}_i^{(n+1)} - \widetilde{y}_{i+1}^{(n)} = (\widetilde{p}_i^{(n+1)} + \widetilde{q}_i^{(n+1)})(\widetilde{x}_i^{(n+1)} - \widetilde{x}_{i+1}^{(n)}),$$

$$\widetilde{p}_i^{(n+1)} - \widetilde{p}_i^{(n)} = (\widetilde{p}_i^{(n+1)} + \widetilde{q}_i^{(n+1)})(\widetilde{q}_i^{(n+1)} - \widetilde{q}_i^{(n)}),$$

$$\widetilde{p}_i^{(n+1)} - \widetilde{p}_{i+1}^{(n)} + \widetilde{q}_i^{(n+1)} - \widetilde{q}_{i+1}^{(n)} = 0,$$

$$\widetilde{u}_i^{(n+1)} = \frac{1}{2}\widetilde{p}_i^{(n+1)}(\widetilde{x}_i^{(n+1)} - \widetilde{x}_i^{(n)}) +$$

$$+\frac{1}{2}\widetilde{q}_i^{(n+1)}(\widetilde{y}_i^{(n+1)} - \widetilde{y}_i^{(n)}) +$$

$$+\frac{1}{2}\widetilde{p}_{i+1}^{(n+1)}(\widetilde{x}_i^{(n+1)} - \widetilde{x}_{i+1}^{(n)}) +$$

$$+\frac{1}{2}\widetilde{q}_{i+1}^{(n+1)}(\widetilde{y}_i^{(n+1)} - \widetilde{y}_{i+1}^{(n)}),$$

$$i, n = 0,1,\ldots,N-1, \quad Nh = a.$$

Theorem 4. Assume that there exists a unique regular solution of the problem (1–3). Then the solution of the above differential scheme $\widetilde{u}$ *converges to the solution u of the problem (1–3) with order* $O(h^2)$*, where* $h = \max(h_1, h_2)$, $h_1 = \max\limits_{0 \le i \le N} |\widetilde{x}_i^{(1)} - \widetilde{x}_i^{(0)}|$, $h_2 = \max\limits_{0 \le i \le N} |\widetilde{y}_i^{(1)} - \widetilde{y}_i^{(0)}|$.

REFERENCES

[1] A.V. Bitsadze, "Some Classes of Partial Differential Equations", *Gordon & Breach Science Pub.*, p. 504, 1988.

[2] J.K. Gvazava, "Nonlocal and initial problems for quasi-linear, non-strictly hyperbolic equations with general solutions represented by superposition of arbitrary functions". *Georgian Math. J.*, **10**, pp. 687–707, 2003.

[3] M.Z.Menteshashvili, "Study of the nonlinear characteristic problem using the Riemann invariants", *Proceedings of A. Razmadze Mathematical Institute, v. 160*, pp. 113-120, 2012.