On the Quantitative Description of Subset Partitions of the Multidimensional Binary Cube

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ABSTRACT

In this paper, the problem of the quantitative description of partitions (QDP) of arbitrary m-subsets of the n-dimensional unit cube is considered for a given $m, 0 \leq m \leq 2^n$. It is shown that QDP can be reduced to the case of those subsets of E^n corresponding to monotone Boolean functions. NP-hardness of the problem is proved.

Keywords

Binary cube, partitions, quantitative description, hypergraphs, degree sequence, monotone Boolean functions.

1. INTRODUCTION

Let $E^n = \{(x_1, x_2, \cdots, x_n) | \, x_i \in \{0,1\}, i = 1, \cdots, n\}$ denote the set of vertices of the n-dimensional unit cube. An arbitrary vertex of E^n is obtained by assigning values to the binary variables $x_1, x_2, \cdots, x_n$. E^n corresponds to the Boolean lattice $(2^{[n]}, \subseteq)$ with the ground set $2^{[n]}$ (the set of all subsets of $[n] = \{1,2, \cdots, n\}$), and with the partial order by inclusion ($\subseteq$). E^n can be visualized through its Hasse diagram [1]. The diagram has $n + 1$ levels numbered from 0 (the lowest level) to n; the k-th level ($k = 0, \cdots, n$) contains all vertices of E^n that have k entries equal to 1. Edges connect those vertices at neighbouring levels that are related by a cover relation. We notice that the k-th and $(n-k)$-th levels contain C_n^k vertices, and they are symmetric with respect to the 0-th and n-th levels for all $k = 0, \cdots, n$.

1.1. Partitioning of E^n

For an arbitrary variable x_i consider the *partitioning/splitting* of E^n into two $(n-1)$-dimensional subcubes of E^n according to the value of x_i:

$E_{x_i=0}^{n-1} = \{(x_1, \cdots, x_n) \in E^n | x_i = 0\}$ and

$E_{x_i=1}^{n-1} = \{(x_1, \cdots, x_n) \in E^n | x_i = 1\}$.

Each set $M \subseteq E^n$ will have (empty or non-empty) subsets in these subcubes: $M_{x_i=1} \subseteq E_{x_i=1}^{n-1}$ and $M_{x_i=0} \subseteq E_{x_i=0}^{n-1}$. Figure 2 shows a schematic picture of the partition.

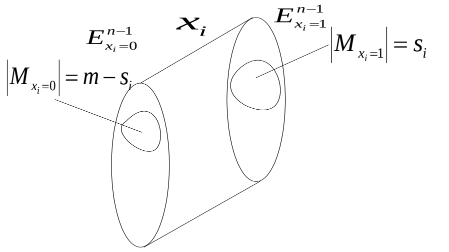

Figure 1. Partitioning of E^naccording to the variable x_i.

1.2. Associated vector of partitions

An integer vector $S = (s_1, \cdots, s_n)$ is called an *associated vector of partitions* of the set $M \subseteq E^n$, if $s_i = |M_{x_i=1}|$ for all $i, i = 1, \cdots, n$.

Associated vector of a set provides a quantitative (numerical) description of the set by means of its partitions. In general, different sets may have the same associated vector of partitions (the same quantitative description of its partitions). For example, the following sets M_1 and M_2 in E^5 have the same associated vector of partitions $S = (4,3,2,2,1)$:

$M_1 = \{(10110), (11001), (11010), (11100)\}$,

$M_2 = \{(10010), (11101), (11000), (11110)\}$.

For a given m, $0 \leq m \leq 2^n$, let $H_m(n)$ denote the set of all m-subsets of E^n, and $D_m(n)$ denote the set of (different) associated vectors of partitions of elements of $H_m(n)$.

In this paper, we consider the following problem:

***Quantitative Description of Partitions (QDP)*:**

given an n-dimensional integer vector d, decide whether d belongs to $D_m(n)$ (in other words, whether d is the associated vector of partitions of an m-subset of E^n).

The paper is organized as follows: Section 2 below brings a necessary condition for the QDP problem by means of minimal and maximal ranks of associated vectors. Then it is shown that the problem can be reduced to the case of those subsets of E^n corresponding to monotone Boolean functions. Section 3 addresses complexity questions of the problem.

2. QUANTITATIVE DESCRIPTION OF SETS' PARTITIONS

2.1. A necessary condition

We define the rank of an element/vector $d = (d_1, \cdots, d_n)$ of $D_m(n)$ as the sum of its components: $r(d) = d_1 + \cdots + d_n$.

Firstly, we find the maximal possible rank of the associated vectors of the elements of $H_m(n)$. For this purpose, we present the integer number m in the following canonical form:

$$m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + \delta,$$

where $0 \leq \delta < C_n^{n-k-1}$, and compose the class $H_{rmax} \subseteq H_m(n)$ in the following way. Every element of H_{rmax} contains all vertices of the n-th, $(n-1)$-th, etc. $(n-k)$-th layers of E^n, and also δ vertices from the $(n-k-1)$-th layer. In this manner, elements of H_{rmax} differ from one another by the choice of δ vertices only. Obviously, the associated vectors of the elements of H_{rmax} have equal ranks, and this is the highest possible rank among all vectors of $D_m(n)$. Denote this rank by r_{max}. r_{max} is calculated as follows ([2]):

$$r_{max} = \sum_{i=0}^{k} (n-i) \cdot C_n^{n-i} + (n-k-1) \cdot \delta.$$

Similarly, the class $H_{rmin} \subseteq H_m(n)$ contains all vertices of the 0-th, 1st, etc., k-th layers of E^n, and also δ vertices from the $(k+1)$-th layer. r_{min}, the smallest possible rank of the associated vectors of elements of $H_m(n)$, can be found by the following formula:

$$r_{min} = \sum_{i=0}^{k} i \cdot C_n^i + (k+1) \cdot \delta.$$

Thus, we obtain a necessary condition for the QDP problem given below by Lemma 1:

Lemma 1. If a given n-dimensional integer vector d belongs to $D_m(n)$, then:

$\sum_{i=0}^{k} i \cdot C_n^i + (k+1) \cdot \delta \le r(d) \le$
$\sum_{i=0}^{k} (n-i) \cdot C_n^{n-i} + (n-k-1) \cdot \delta.$

To evaluate this condition we need additional analysis and estimations on the difference between r_{max} and r_{min}, the number of elements of $D_m(n)$ depending on m, the number of vectors of a given rank, etc., - which is a subject of our further research. Some preliminary distances and estimates of the mentioned type are given in [3].

2.2. Problem reduction (using subset exchange procedures)

First we introduce the <u>exchange (or shifting) operations</u>: $S_i^{0\text{-}1}$ and $S_i^{1\text{-}0}$ in E^n.

Let $\mathcal{M} \subseteq E^n$; $\tilde{\alpha}$ and $\tilde{\alpha}'$ be two vertices of E^n, such that $\tilde{\alpha}$ belongs to the given subset $\mathcal{M}$ of E^n, and $\tilde{\alpha}' \notin \mathcal{M}$.

The <u>*vertex exchange operation*</u> $S_i^{0\text{-}1}(\tilde{\alpha})$ performs the following:

- If $\alpha_i = 0$ in $\tilde{\alpha}$, and $\alpha_i' = 1$ in $\tilde{\alpha}'$, and $\alpha_k' = \alpha_k$ for all $k \ne i$, then $\tilde{\alpha} \in \mathcal{M}$ is replaced with $\tilde{\alpha}' \in E^n$;
- Otherwise, $\tilde{\alpha}$ remains unchanged.

$S_i^{1\text{-}0}(\tilde{\alpha})$ *operation* performs the following:

- If $\alpha_i = 1$ in $\tilde{\alpha}$, and $\alpha_i' = 0$ in $\tilde{\alpha}'$, and $\alpha_k' = \alpha_k$ for all $k \ne i$, then $\tilde{\alpha}$ is replaced with $\tilde{\alpha}'$;
- Otherwise, $\tilde{\alpha}$ remains unchanged.

Let $\mathcal{M}$ be an m-subset of vertices in E^n; and $\mathcal{M}_{x_i=1}$ and $\mathcal{M}_{x_i=0}$ be corresponding subsets of $\mathcal{M}$ in $E_{x_i=1}^{n-1}$ and $E_{x_i=0}^{n-1}$.

Operation $S_i^{0\text{-}1}(\mathcal{M})$ performs vertex exchange operations $S_i^{0\text{-}1}(\tilde{\alpha})$ over all vertices $\tilde{\alpha}$ of $\mathcal{M}_{x_i=0}$, and *operation* $S_i^{1\text{-}0}(\mathcal{M})$ performs vertex exchange operations $S_i^{1\text{-}0}(\tilde{\alpha})$ over all vertices $\tilde{\alpha}$ of $\mathcal{M}_{x_i=1}$.

$S_i^{0\text{-}1}(\mathcal{M})$ and $S_i^{0\text{-}1}(\mathcal{M})$ will be referred to as <u>*subset exchange operations*</u> *over the set* $\mathcal{M}$ in the direction i. These operations may be applied individually, as well as concurrently because of their domains are none intersecting.

Suppose that $S_i^{0\text{-}1}(\mathcal{M})$ and $S_i^{1\text{-}0}(\mathcal{M})$ are applied over the set $\mathcal{M}$, and let $\bar{\mathcal{M}}$ denote the resulting set in E^n. Thus: when $S_i^{0\text{-}1}(\mathcal{M})$ is applied, then $|\bar{\mathcal{M}}_{x_i=0}| \le |\mathcal{M}_{x_i=0}|$, $|\mathcal{M}_{x_i=1}| \le |\bar{\mathcal{M}}_{x_i=1}|$ and $|\mathcal{M}_{x_i=0}| - |\bar{\mathcal{M}}_{x_i=0}| = |\bar{\mathcal{M}}_{x_i=1}| - |\mathcal{M}_{x_i=1}|$. Similar relations are valid for the operation $S_i^{1\text{-}0}(\mathcal{M})$. In terms of subset ranks we see that the operation $S_i^{0\text{-}1}(\mathcal{M})$ implies the rank increase and $S_i^{1\text{-}0}(\mathcal{M})$ implies the rank decrease. Consecutive implementation in different directions, repeating directions many times leads to monotone Boolean functions for $S_i^{0\text{-}1}(\mathcal{M})$, and to the negations of these functions for $S_i^{1\text{-}0}(\mathcal{M})$.

When $S_i^{0\text{-}1}(\mathcal{M})$ and $S_i^{1\text{-}0}(\mathcal{M})$ are applied concurrently, then $|\mathcal{M}_{x_i=0}|=|\bar{\mathcal{M}}_{x_i=1}|$, and $|\mathcal{M}_{x_i=1}|=|\bar{\mathcal{M}}_{x_i=0}|$. If $|\mathcal{M}_{x_i=1}| = d_i$ then $|\bar{\mathcal{M}}_{x_i=1}| = m - d_i$, and vice versa. d_i and $m - d_i$ in this context are called *inversions* of each other in the direction i and with respect to the size m.

Now we define the following relation on the set of all m-subsets $H_m(n)$, denoted by $\mathcal{R}_1$:

$H_1\mathcal{R}_1H_2$ (H_1 is in relation $\mathcal{R}_1$ with H_2) if and only if H_2 can be obtained from H_1 by k, $k \in \{0,1,\cdots,n\}$ subset exchange operations in several directions, where H_1 and H_2 are elements of $H_m(n)$.

It is easy to check that the relation $\mathcal{R}_1$ obeys the reflexivity, symmetricity and transitivity properties, and thus, it is an *equivalence relation* over $H_m(n)$. In this manner, "subset exchange" splits $H_m(n)$ into disjoint sets /equivalence classes/ $\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_p$, which cover the whole $H_m(n)$.

Relation $\mathcal{R}_1$ is defined over the set $H_m(n)$ of all m-subsets of E^n. Similarly, we can define "inversion" relation over the set of all association vectors $D_m(n)$, denoting it by $\mathcal{R}_2$:

$D_1\mathcal{R}_2D_2$ (D_1 is in relation $\mathcal{R}_2$ with D_2) if and only if D_1 can be obtained from D_2 by k ($k = 0,1,\cdots,n$) number of inversions in different directions, where D_1 and D_2 are arbitrary elements of $D_m(n)$.

$\mathcal{R}_2$ is an *equivalence relation* on $D_m(n)$, and thus, it splits $D_m(n)$ into disjoint sets/equivalence classes $\tilde{D}_{i_1}, \tilde{D}_{i_2}, \cdots, \tilde{D}_{i_q}$, which cover the whole $D_m(n)$.

We note that $\tilde{D}_{i_1}, \tilde{D}_{i_2}, \cdots, \tilde{D}_{i_q}$ are not arbitrary classes of $D_m(n)$, they come from $\tilde{H}_1, \tilde{H}_2, \cdots, \tilde{H}_p$ in the following way. Let $\tilde{D}_j$ denote the class of associated vectors of partitions of the elements of $\tilde{H}_j$, $j = 1, \cdots p$. There may be coinciding classes among $\tilde{D}_j$ (since different sets may have the same associated vector of partitions). Removing the repeated ones, we will obtain equivalence classes $\tilde{D}_{i_1}, \tilde{D}_{i_2}, \cdots, \tilde{D}_{i_q}$ ($q \le p$).

Let $d = (d_1, d_2, \cdots, d_n)$ belong to some equivalence class $\tilde{D}_{i_j}$. It is easy to check that $|\tilde{D}_{i_j}| = 2^k$, where $k = |\{d_i | d_i \ne (m - d_i)\}|$. Obviously, $k = n$ for odd m. Additional quantitative relations are given in [4].

Consider an example to interpret the equivalence classes.

Example 1.

Let $n = 3, m = 4$. Then $H_4(3)$, the set of all 4-subsets of E^3 consists of $C_{2^3}^4 = 70$ elements.

There are 11 equivalence classes (according to the relation $\mathcal{R}_2$) covering $D_4(3)$, those are:

$\tilde{D}_{i_1} = \{(3,3,3), (1,3,3), (3,1,3), (3,3,1), (1,1,3),$
$(1,3,1), (3,1,1), (1,1,1)\}, |\tilde{D}_{i_1}| = 8$
$\tilde{D}_{i_2} = \{(3,3,2), (1,3,2), (3,1,2), (1,1,2)\}, |\tilde{D}_{i_2}| = 4$
$\tilde{D}_{i_3} = \{(3,2,3), (1,2,3), (3,2,1), (1,2,1)\}, |\tilde{D}_{i_3}| = 4$
$\tilde{D}_{i_4} = \{(2,3,3), (2,1,3), (2,3,1), (2,1,1)\}, |\tilde{D}_{i_4}| = 4$
$\tilde{D}_{i_5} = \{(3,2,2), (1,2,2)\}, |\tilde{D}_{i_5}| = 2$
$\tilde{D}_{i_6} = \{(2,3,2), (2,1,2)\}, |\tilde{D}_{i_6}| = 2$
$\tilde{D}_{i_7} = \{(2,2,3), (2,2,1)\}, |\tilde{D}_{i_7}| = 2$
$\tilde{D}_{i_8} = \{(4,2,2), (0,2,2)\}, |\tilde{D}_{i_8}| = 2$
$\tilde{D}_{i_9} = \{(2,4,2), (2,0,2)\}, |\tilde{D}_{i_9}| = 2$
$\tilde{D}_{i_{10}} = \{(2,2,4), (2,2,0)\}, |\tilde{D}_{i_{10}}| = 2$
$\tilde{D}_{i_{11}} = \{(2,2,2)\}, |\tilde{D}_{i_{11}}| = 1.$

Though, the number of equivalence classes $\tilde{H}_j$ are much more. For example, the same class $\tilde{D}_{i_{11}} = \{(2,2,2)\}$ corresponds to different $\tilde{H}_j$ classes (in this case each of these classes consists of the unique set); those are:

$\{(000), (001), (110), (111)\}$,
$\{(000), (011), (101), (111)\}$,
$\{(000), (011), (100), (111)\}$,
$\{(000), (011), (101), (110)\}$,
$\{(001), (010), (100), (111)\}$,
$\{(001), (010), (101), (110)\}$,
$\{(001), (010), (101), (110)\}$.

For every class $\tilde{D}_{i_j}$ if we find any element of the class, then we will obtain also all its elements. It remains only to identify /to find/ a certain element, and instead of the entire class, consider that one.

Notice that every class $\tilde{D}_{i_j}$ contains a unique vector with all $\ge \lceil m/2 \rceil$ components, and a unique vector with all $\le \lfloor m/2 \rfloor$ components. Each of these vectors can play the role of so called "*generating element*" for the whole class $\tilde{D}_{i_j}$. Therefore, we can restrict attention with the vectors with all $\ge \lceil m/2 \rceil$ components or with all $\le \lfloor m/2 \rfloor$ components.

Without loss of generality, we will consider the case of all $\geq \lceil m/2 \rceil$ components; further we will refer to them as *generating elements*.
In Example 1 the generating elements are:
(3,3,3) for $\widetilde{D}_{i_1}$,
(3,3,2) for $\widetilde{D}_{i_2}$, (3,2,3) for $\widetilde{D}_{i_3}$, (2,3,3) for $\widetilde{D}_{i_4}$,
(3,2,2) for $\widetilde{D}_{i_5}$, (2,3,2) for $\widetilde{D}_{i_6}$, (2,2,3) for $\widetilde{D}_{i_7}$,
(4,2,2) for $\widetilde{D}_{i_8}$, (2,4,2) for $\widetilde{D}_{i_9}$, (2,2,4) for $\widetilde{D}_{i_{10}}$
(2,2,2) for $\widetilde{D}_{i_{11}}$.
Let $\overline{D}_m(n)$ denote the set of generating elements of all $\widetilde{D}_{i_j}$ classes (in other words, a subset of $D_m(n)$ consisting of all vectors with all $\geq \lceil m/2 \rceil$ components). In these terms, the problem QDP can be reduced to the following problem:
***Reduced Quantitative Description* (*RQD*):**
Given n-dimensional integer vector $\bar{d}$ with all $\geq \lceil m/2 \rceil$ components; determine whether $\bar{d}$ belongs to $\overline{D}_m(n)$ (in other words, is $\bar{d}$ generating element for some equivalency class $\widetilde{D}_{i_j}$)?

2.3. Further reduction of the problem (using monotone Boolean functions)

In this section, we consider further reduction of the problem and prove that among all generating elements it suffices to find only those corresponding to monotone Boolean functions.
$f(x_1, x_2, \cdots, x_n): E^n \to \{0,1\}$ function is called a *Boolean function*.
Define a component-wise partial order on E^n: $(\alpha_1, \cdots, \alpha_n) \preccurlyeq (\beta_1, \cdots, \beta_n)$ (vertex $(\alpha_1, \cdots, \alpha_n)$ precedes vertex $(\beta_1, \cdots, \beta_n)$) if $\alpha_i \leq \beta_i$ for $i = 1, \cdots, n$.
Vertex $(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j = 1, \alpha_{j+1}, \cdots, \alpha_n)$ of E^n is called an *upper neighbor* of the vertex $(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j = 0, \alpha_{j+1}, \cdots, \alpha_n)$ by the j-th direction.
Boolean function $f(x_1, x_2, \cdots, x_n)$ is *monotone* if for arbitrary $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_n)$ vertices of E^n, $(\alpha_1, \cdots, \alpha_n) \preccurlyeq (\beta_1, \cdots, \beta_n)$ implies $f(\alpha_1, \cdots, \alpha_n) \leq f(\beta_1, \cdots, \beta_n)$.
$\alpha^1 \in E^n$ is called a *lower unit* of f, if $f(\alpha^1) = 1$, and $f(\alpha) = 0$ for all α with $\alpha \preccurlyeq \alpha^1$. $\alpha^0 \in E^n$ is called an *upper zero* of f, if $f(\alpha^0) = 0$, and $f(\alpha) = 1$ for all α with $\alpha^0 \preccurlyeq \alpha$.
It follows from the definition that if a monotone Boolean function accepts the value "1" on some vertex $(\alpha_1, \cdots, \alpha_n)$ of E^n, then it accepts the value "1" also on all upper neighbors of $(\alpha_1, \cdots, \alpha_n)$.
Each Boolean function can be presented geometrically on the n-dimensional unit cube by distinguishing those vertices of E^n, in which the function accepts the value "1". Thus, monotone Boolean functions compose a specific class of subsets in E^n; let $\widehat{M}_m(n)$ denote the set of all m-subsets of E^n corresponding to monotone Boolean functions, which have m values "1"; and let $D_m^{\widehat{M}}(n)$ denote the set of associated vectors of their elements.
The following lemma states that $D_m^{\widehat{M}}(n) \subseteq \overline{D}_m(n)$.
Lemma 2. If $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_n) \in D_m^{\widehat{M}}(n)$ then $\hat{d}_i \geq \lceil m/2 \rceil$ for $i = 1, \cdots, n$.
Proof. Let $\hat{d} = (\hat{d}_1, \cdots, \hat{d}_n) \in D_m^{\widehat{M}}(n)$ and $\widehat{M}$ be a set from $\widehat{M}_m(n)$ the associated vector of which is $\hat{d}$. Suppose by a contradictory assumption that $\hat{d}_j < \lceil m/2 \rceil$ for some j. Then there must be a vertex $(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j = 0, \alpha_{j+1}, \cdots, \alpha_n)$ in $\widehat{M}$ such that the vertex $(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j = 1, \alpha_{j+1}, \cdots, \alpha_n)$ does not belong to $\widehat{M}$. But in this case $\widehat{M}$ cannot be the set of values "1" of any monotone Boolean function. Thus, $\hat{d}_j \geq \lceil m/2 \rceil$. □

Theorem 1. Given an n-dimensional integer vector $\bar{d}$ with all components $\geq \lceil m/2 \rceil$. $\bar{d}$ belongs to $\overline{D}_m(n)$ if and only if there exists $\hat{d} \in D_m^{\widehat{M}}(n)$ such that $\bar{d} \leq \hat{d}$.
Proof.
Necessity
Suppose that $\bar{d}$ is a vector satisfying the theorem condition, and $\bar{d} \in \overline{D}_m(n)$. If $\bar{d} \in D_m^{\widehat{M}}(n)$, then the necessity part is proved. Now, let us assume that $\bar{d} \notin D_m^{\widehat{M}}(n)$; it follows that M, a corresponding to $\bar{d}$ set in E^n, does not belong to $\widehat{M}_m(n)$. Applying $S_i^{0\text{-}1}(M)$ exchange operations consecutively for all directions i, will lead to a set $\widehat{M}$ from $\widehat{M}_m(n)$ with an associated vector $\hat{d}$ greater than $\bar{d}$.
Sufficiency
Suppose that $\bar{d}$ is a vector satisfying the theorem condition, and there exists $\hat{d} \in D_m^{\widehat{M}}(n)$ such that $\bar{d} \leq \hat{d}$. Let $\widehat{M}$ denote a corresponding to $\hat{d}$ set from $\widehat{M}_m(n)$. We have to prove that $\bar{d}$ is the associated vector for some m-subset of E^n. Suppose that $\bar{d}_j < \hat{d}_j$ for some j. Split E^n according to the variable x_j ($\widehat{M}_{x_j=1}$ and $\widehat{M}_{x_j=0}$ will denote the corresponding subsets of $\widehat{M}$). We need to prove that there are at least $\hat{d}_j - \bar{d}_j$ vertices in $\widehat{M}_{x_j=1}$, which can be moved into $\widehat{M}_{x_j=0}$ by applying $S_j^{1\text{-}0}$ vertex exchange operations. Suppose by a contradictory assumption that the number of such vertices is k, and $k < \hat{d}_j - \bar{d}_j$. It follows that $\hat{d}_j - k = \lfloor m/2 \rfloor$. Thus, on the one hand, $\bar{d}_j < \hat{d}_j - k$, and on the other hand, $\lceil m/2 \rceil \leq \bar{d}_j$, and then: $\lceil m/2 \rceil < \hat{d}_j - k$. This contradiction completes the proof. □
Theorem 1 implies that the problem RQD can be reduced to the following problem:
Monotone Quantitative Description (MQD):
Given n-dimensional integer vector $\bar{d}$ with all $\geq \lceil m/2 \rceil$ components; does there exist $\hat{d}$ from $D_m^{\widehat{M}}(n)$ such that $\bar{d} \leq \hat{d}$?
Consider again the Example 1.
Among all generating elements only the following vectors correspond to monotone Boolean functions:
(3,3,3) for $\widetilde{D}_{i_1}$,
(4,2,2) for $\widetilde{D}_{i_8}$, (2,4,2) for $\widetilde{D}_{i_9}$, (2,2,4) for $\widetilde{D}_{i_{10}}$.
Additionally, we can take into account that vectors (4,2,2), (2,4,2), and (2,2,4) correspond to the isomorphic monotone Boolean functions, which have the following sets of "1"s:
$M_1 = \{(100), (101), (110), (111)\}$,
$M_2 = \{(010), (011), (110), (111)\}$,
$M_3 = \{(001), (011), (101), (111)\}$,
and thus, it suffices to consider only one of those vectors.
Concluding, - to obtain the associated vectors for all 70 sets of $H_4(3)$, it is sufficient to find only two associated vectors: (3,3,3) and (4,2,2).

Concluding this section, it is worth mentioning the following findings of Theorem 1.
Let an integer number m be presented in the following form: $m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_p}$, where $k_1 > k_2 > \cdots > k_p \geq 0$.
It follows that the lowest layer of E^n that can contain lower units of a monotone Boolean function from $\widehat{M}_m(n)$, is the $(n - k_1)$-th layer. In this manner, instead of looking for an m-subset of vertices thorough the whole E^n, it suffices to look for an m-subset of vertices on the $(n - k_1)$-th and higher layers.

3. COMPLEXITY OF THE QUANTITATIVE DESCRIPTION

3.1. Relation with hypergraphs

Let $[n] = \{1,2, \cdots, n\}$. Consider the power set of $[n]$ (denoted by $\mathcal{P}([n])$) and its partial order by inclusion: let a and b be arbitrary subsets of $[n]$; a precedes b, if $a \subseteq b$. Identify subsets of $[n]$ with binary vectors of length n such that the i-th entry equals "1" if and only if the i-th element of $[n]$ is included in the subset. In this manner, 1-1 correspondence between $\mathcal{P}([n])$ and E^n is established. Each $\mathcal{M} \subseteq E^n$ can be identified with an element of $\mathcal{P}([n])$, or in other words, with a *simple hypergraph*[1] $\mathcal{H}$ on vertex set $[n]$, the edges of which determined by the elements of $\mathcal{M}$. Then the *degree* of the i-th vertex of $\mathcal{H}$ is equal to $|\mathcal{M}_{x_i=1}|$; the associated vector of partitions of $\mathcal{M}$ corresponds to the degree sequence of $\mathcal{H}$. $H_m(n)$ will correspond to the set of all simple hypergraphs, with vertex set $[n] = \{1,2,\cdots,n\}$ and with m hyperedges, and $D_m(n)$ will be the set of all degree sequences of elements of $H_m(n)$. The QDP problem in terms of hypergraphs is equivalent to the *hypergraph degree sequence problem:* does there exist a simple hypergraph by a given degree sequence? This is a well-known open problem in the graph theory, stated in [5]. Recently the case of k-uniform hypergraphs has been solved - it is proved that the existence problem for simple 3-uniform hypergraphs is NP-complete [6].

3.2. Complexity

In this section, we prove that the QDP problem is NP-hard.

Recall the class H_{rmax} of m-subsets of E^n (defined in Section 2) and calculate the components of associated vectors of elements of H_{rmax}. Each component can be presented as a sum of two summands. The first summand is constant for each component and comes from the vertices of the n-th, $(n-1)$-th, etc., $(n-k)$-th layers of E^n, and is equals to ([2]):

$$\sum_{j=0}^{k} \frac{(n-j)\cdot C_n^{n-j}}{n} = \sum_{j=0}^{k} \frac{(n-j)\cdot n!}{n\cdot(n-j)!\cdot j!} = \sum_{j=0}^{k} C_{n-1}^{n-j-1}.$$

The second summand comes from δ vertices of the C_n^{n-k-1} layer. If denote by s_i the value of the i-th component, then $(s_1,\cdots,s_n)$ will correspond to the associated vector of partitions of δ-set from the $(n-k-1)$-th layer of E^n.

Thus, the following statement holds:

Theorem 2. Let $m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + \delta$, $0 \le \delta < C_n^{n-k-1}$. An n-dimensional integer vector $(d_1,\cdots,d_n)$ is the associated vector of partitions for some set of H_{rmax} (for the given m) if and only if $(s_1,\cdots,s_n)$ is the associated vector for some δ-set of vertices from the $(n-k-1)$-th layer of E^n, where $s_i = d_i - \sum_{j=0}^{k} C_{n-1}^{n-j-1}$ for all $i = 1,\cdots,n$.

For the case of $k = 1$ we formulate the following statement:

Theorem 3. Let $m = C_n^n + C_n^{n-1} + \delta$, where $0 < \delta \le C_n^{n-2}$. An n-dimensional integer vector $(d_1,\cdots,d_n)$ is the associated vector of partitions for some set of H_{rmax} (for the given m) if and only if each d_i can be presented as: $d_i = n + s_i$, where $\sum_{1=1}^{n} s_i = (n-2)\delta$, and $(\delta - s_1,\cdots,\delta - s_n)$ permutated in decreasing order: $\delta - s_{t_1} \ge \cdots \ge \delta - s_{t_n}$, satisfies the following condition (Erdos-Gallai inequalities):

$$\sum_{i=1}^{j} (\delta - s_{t_i}) - \sum_{i=l+1}^{n} (\delta - s_{t_i}) \le j(l-1),\ 1 \le j \le l \le n.$$

Proof follows from Theorem 2 and from the Erdos-Gallai theorem ([7], [8]). □

Note that deciding whether a given vector $(s_1,\cdots,s_n)$ is the associated vector for some δ-set of vertices from the given layer of E^n is equivalent to the hypergraph degree sequence problem for uniform hypergraphs. Therefore, it follows from Theorem 2 and Theorem 3 that the problem of deciding whether a given n-dimensional integer vector $(d_1,\cdots,d_n)$ is an associated vector for some set of H_{rmax} is NP-complete for $m > C_n^n + C_n^{n-1} + C_n^{n-2}$, and belongs to the complexity class P for $m \le C_n^n + C_n^{n-1} + C_n^{n-2}$.

Theorem 4. Let $m = C_n^n + C_n^{n-1} + \cdots + C_n^{n-k} + \delta$, $0 < \delta \le C_n^{n-k-1}$. The problem QDP is NP-hard for $k \ge 2$.

Proof.

The input of the problem QDP is an n-dimensional integer vector $(d_1,\cdots,d_n)$. A solution/certificate for QDP can be an m-subset of vertices of E^n (where each vertex is an n-dimensional binary vector). Since m is an arbitrary integer less than 2^n, we cannot provide that the solution can be checked in polynomial time.

Now we prove that $(n-k-1)$-uniform hypergraph degree sequence problem can be reduced to QDP in polynomial time. First we notice that H_{rmax} is the only class of m-subsets with the associated vectors of the rank $r_{max} = \sum_{i=0}^{k}(n-i)\cdot C_n^{n-i} + (n-k-1)\cdot\delta$. Therefore, the problem of deciding whether a given n-dimensional integer vector is an associated vector for some set of H_{rmax}, can be considered as a particular case of QDP, formulated in the following way: given an n-dimensional integer vector d such that the sum of its components is equal to r_{max}; decide whether d belongs to $D_m(n)$.

On the other hand it follows from Theorem 2 that the problem of deciding whether the given n-dimensional integer vector is an associated vector for some set from the given layer of E^n is also a particular case of QDP. But this is equivalent to the $(n-k-1)$-uniform hypergraph degree sequence problem, which completes the proof. □

4. ACKNOWLEDGEMENT

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[1] Let $V = \{v_1,\cdots,v_n\}$ be a finite set. A *hypergraph* $H = \{E_1,\cdots,E_m\}$ on V is a family of subsets $E_i \subseteq V$, $i = 1,\cdots,m$. $v_1,\cdots,v_n$ are called *vertices* of the hypergraph, and $E_1,\cdots,E_m$ are *hyper-edges*. Hypergraph is *r-uniform* if each of its hyper-edges contains exactly r elements. Hypergraph is *simple* if all hyper-edges are different. *Degree* d_j of vertex $v_j \in V$ is the number of hyperedges containing v_j. The sequence $d(H) = \{d_1,\cdots,d_n\}$ consisting of degrees of vertices of hypergraph H is called a *hypergraph degree sequence.*