About the Spectrum of the Eigenvalues of Color Operators in a Theory of Canonically Conjugate Fuzzy Subsets

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ABSTRACT

In the present work is considered an approach, according to which canonically conjugate colors in the theory of fuzzy sets are related to the properties of information functions and non-commutative linear operators in Gilbert's space: each information state corresponds to the estimation of compatibility function, every color – to the operator.
It is supposed that color, as some property (attribute), characterizing a condition of a system, can receive different values called by eigenvalues of this color. The cases of discrete and continuous spectrum of eigenvalues of color are considered. The example of calculation of conditional computable values of color is given.

Keywords

Information functions, canonically conjugate fuzzy subsets, color operators.

1. INTRODUCTION

In the theory of canonically conjugate fuzzy subsets, the main ideas of which are formulated in [1-5], the main concept is a notation of “color” (property, attribute), characterizing the state of an observed system. Under the system state is implied the collection of data, which is a result of experts’ (experimenter who conducts measurements) activities. The idea of introducing this notation, as well as notation of canonically conjugate fuzzy subsets, belongs to T.Gachechiladze [1].
It’s important to notice that these notations and the quantum mechanics method of their consideration directly lead to the theory of color (the new chromo-theory of canonically conjugate fuzzy subsets).
In the present paper is considered the approach given in [1], according to which the theory of color, in its usual form, deals with properties of information functions and the corresponding operators in a Hilbert space, so that each informational state corresponds to the estimation of compatibility function, each color – to the operator.
Suppose that universal set Ω with elements ω and defined on it the certain attribute (property) $\wp$ with numerical characteristic $\xi_\wp$ is given. It is supposed that $\xi_\wp$ is a random quantity, distribution of probabilities $\xi_\wp \in R$ of which is characterized by the density $\rho_\wp(x_\omega)$.
Suppose then that to each element of Ω corresponds the color $\wp$ and we can prescribe to any $\omega \in \Omega$ the measure of its compatibility $\mu_\wp(\omega)$ with the color $\wp$. It’s important to notice that this function depends, besides of ω, from the observer as well, who performs the estimation of color $\wp$. Therefore, the role of observer is reduced to the estimation of color $\wp$ with a certain accuracy.
Following to [1], the information, accessible to the observer, as known from [6] is involved in the function $\psi_\wp(x_\omega) \in L^2(R)$ and

$$\rho_\wp(x_\omega) = \left|\psi_\wp(x_\omega)\right|^2,\ \omega \in \Omega. \quad (1)$$

From (1) it follows that together with color $\wp$ there exists another color $\wp^c$, characterized fully,

$$\rho_{\wp^c}(x_{c\omega}) = \left|\psi_{\wp^c}(x_{c\omega})\right|^2,\ \omega \in \Omega, \quad (2)$$

where

$$\psi_{\wp^c}(x_{c\omega}) = \hat{F}\psi_\wp(x_\omega) = \frac{1}{\sqrt{2\pi c}}\int_R \psi_\wp(x_\omega) e^{-\frac{i}{c}x_{c\omega}x_\omega} dx_\omega \quad (3)$$

is a Fourier transformation of the function $\psi_\wp(x_\omega)$ and c is a constant.
The color $\wp^c$ is called canonically conjugate with respect to $\wp$ [1]. As a result, we can conclude that along with fuzzy subset $\tilde{\Omega}$ there exists its canonically conjugate subset $\tilde{\Omega}^c$. Moreover, the existence of canonically conjugate colors $\wp$ and $\wp^c$ of $\omega \in \Omega$ is characterized by the conditional instances (moments) $\left(M_{\xi_\wp}, \sigma_\wp^2\right)$ and $\left(M_{\xi_{\wp^c}}, \sigma_{\wp^c}^2\right)$ of mathematical expectation and dispersion of random variables $\xi_\wp(\omega)$ and $\xi_{\wp^c}$, respectively, where $\xi_{\wp^c}$ is a numerical characteristic of the color $\wp^c$. Between canonically conjugate colors $\wp$ and $\wp^c$ is close connection. For its clarification one can correspond them noncommutative linear operators, mapping the Hilbert space of functions $\psi_\wp(x_\omega)$ into itself [6].

2. COLOR OPERATOR

Let’s consider some attribute (color) $\wp$ characterizing the system state. Under the system state is implied the set of data, which is a result of experimenters’ or experts’ activities. For example, expert study the compatibility function (that is define information function). Strictly speaking, in our judgment we should talk not about one color, but about many colors instantaneously. However, this doesn't change the reasoning and for brevity and simplicity we will consider only one color below. Values, that can take one color are named as eigenvalues and their set - as a spectrum of eigenvalues of color $\wp$. The spectrum, evidently, may be discrete or continuous.

At first, we suppose that spectrum of color $\wp$ is discrete. Eigenvalues are denoted by $x_{\omega n}(n \in N)$. Using Dirac's bracket notations, denote by $|x_{\omega n};\ \wp\rangle$ an information function of the system in a state when color $\wp$ has a value $x_{\omega n}$. Remark that under the information function of color $\wp$ we consider the following expression:

$$|x_{\omega n};\ \wp\rangle = \sqrt{\rho_{\wp}(x_{\omega n})}e^{i\varphi_{\wp}}, \tag{4}$$

where real φ is an arbitrary phase.

These informational functions are named as eigenfunctions of the given color $\wp$. Each of these functions are normed:

$$\int_R \left||x_{\omega n};\ \wp\rangle\right|^2 dx_{\omega} = 1.$$

It is not necessary that the system is in some "eigen"-state with information function $|x_{\omega n};\ \wp\rangle$. The expert creates the state (estimating the values of $\wp$) as a result of his intellectual activity, gives one of the eigenvalues (makes decision that the value of color is equal to $x_{\omega n}$). It is necessary to mention that as the intervals, corresponded to different colors, generally are overlapping, thus this fact should be considered in a model. One of the possibilities consists of letting the superposition principle to be valid for information functions. Regarding this principle the information function $|x_{\omega};\ \wp\rangle$ must be a linear combination of functions $|x_{\omega n};\ \wp\rangle$ corresponding to the values $x_{\omega n}$, which can be observed with different from zero compatibility, when the expert is making the estimation of system state.

Thus, in case of general state, the information function $|x_{\omega};\ \wp\rangle$ can be expanded in the series:

$$|x_{\omega};\ \wp\rangle = \textstyle\sum_n a_n |x_{\omega n};\ \wp\rangle, \tag{5}$$

where summation is performed by all n and a_n are some constants coefficients.

Thus, we conclude that any information function can be expanded by eigenfunctions of arbitrary color. The function system, by which any information function can be expanded, is a full system of information functions.

The mathematical model reflects the influence of measurements or expert estimation on a state (collection of data) of a system and permits to calculate mean value of estimated color $\wp$ of an object in any state $|x_{\omega};\ \wp\rangle$. In it the correspondence between the color $\wp$ and the linear Hermitian operator $\hat{\wp}$ is given. Eigenvectors of this operator are state vectors $|x_{\omega n};\ \wp\rangle$, in which $\wp$ takes a certain value $x_{\omega n}$. They are the eigenvalues of this operator:

$$\hat{\wp}|x_{\omega n};\ \wp\rangle = x_{\omega n}\,|x_{\omega n};\ \wp\rangle. \tag{6}$$

Such definition can be made because the eigenvectors of linear Hermitian operator form the system of orthogonal normed vectors. The representation of observed color $\wp$ by linear operator $\hat{\wp}$, which satisfies the equation (6) is convenient also because the operator $\hat{\wp}$ transforms the state vector $|x_{\omega};\ \wp\rangle$ in another vector $|x'_{\omega};\ \wp\rangle$,

$$\hat{\wp}|x_{\omega};\ \wp\rangle = |x'_{\omega};\ \wp\rangle \tag{7}$$

in such a way that the projection of the vector $|x'_{\omega};\ \wp\rangle$ on $|x_{\omega};\ \wp\rangle$ is a mathematical expectation in the state $|x_{\omega};\ \wp\rangle$:

$$x^*_{\omega} \equiv M\xi_{\wp} = \langle\hat{\wp}\rangle = \langle x_{\omega};\ \wp|x'_{\omega};\ \wp\rangle = \langle\omega;\wp|\hat{\wp}|x_{\omega};\ \wp\rangle. \tag{8}$$

Analogously,

$$\hat{\wp}^c|x_{c\omega};\wp^c\rangle = x_{c\omega}|x_{c\omega};\wp^c\rangle, \tag{9}$$
$$\wp^c|x_{c\omega};\wp^c\rangle = |x'_{c\omega};\wp^c\rangle,$$

and

$$x^*_{c\omega} \equiv \langle\hat{\wp}^c\rangle = \hat{M}\hat{F}\xi_{\wp^c}(\omega^c) = \langle x_{c\omega};\ \wp^c|x'_{c\omega}\rangle =$$
$$= \langle x_{c\omega};\ \wp^c|x_{c\omega};\ \wp^c\rangle.$$

It is known [6] that the operators $\hat{\wp}$ and $\hat{\wp}^c$ are related with the following commutative relationship:

$$\hat{\wp}\hat{\wp}^c - \hat{\wp}^c\hat{\wp} = ic\hat{E},$$

where $\hat{E}$ is an operator of identical transformation. This relationship defines the quantitative relation between canonically conjugate colors and limits their simultaneous "measurability". If we follow to Weyl, according to [1], the relation between the canonically conjugate colors $\wp$ and $\hat{\wp}$ is defined in a form of the principle of uncertainty for dispersions $\sigma^2_{\wp}$ and $\sigma^2_{\wp^c}$ of canonically conjugate colors [2,3]:

$$\sigma^2_{\wp}\,\sigma^2_{\wp^c} \geq \frac{c^2}{4}.$$

Suppose, $\xi_{\wp}$ is a value of some attribute $\wp$ with continuous spectrum. Its eigenvalues will be denoted by $x_{\wp}(\omega)$ and corresponding eigenfunctions – by $|x_{\wp}(\omega);\ \wp\rangle$. As any information function of $\wp$ with discrete spectrum can be decomposed in series (5), information function with continuous spectrum can be decomposed in integral. Such decomposition has the following form:

$$|x_{\omega};\ \wp\rangle = \int_R a_{\wp}(x'_{\omega})|x'_{\wp}(\omega);\ \wp\rangle dx'_{\wp}(\omega). \tag{10}$$

In case of the continuous spectrum, a special consideration is necessary for the question of information function's normalization, because the equality to one of the integral from module square of information function cannot be satisfied. Instead we will normalize the function $|x_{\wp}(\omega);\ \wp\rangle$ in such a way that $|a_{\wp}(\omega)|^2$ represents the probability of color $\wp$ in interval $x_{\wp}(\omega)$ and $x_{\wp}(\omega) + dx_{\wp}(\omega)$. Because the sum of all values of $\xi_{\wp}(\omega)$ will be equal to one, then

$$\int_R |a_{\wp}(\omega)|^2 dx_{\omega} = 1 \text{ (compare with } \textstyle\sum_n |a_n|^2 = 1). \tag{11}$$

From the last formula it is clear, that integral in (11) must be an expression, bilinear with respect to $|x_{\omega};\ \wp\rangle$ and $|x_{\omega};\ \wp\rangle^+ = \langle x_{\omega};\wp|$. It must be equal to 1 for appropriate normalization of $|x_{\wp}(\omega);\ \wp\rangle$. Thus, in discrete case we must have the following equality:

$$\textstyle\sum_n a^*_n a_n = \int_R \langle x_{\omega};\wp|x_{\omega};\ \wp\rangle dx_{\omega}.$$

Similarly, in continuous case we will have:

$$\int_R |a_{\wp}(\omega)|^2 dx_{\omega} = \int_R \langle x_{\omega};\wp|x_{\omega};\ \wp\rangle dx_{\omega}. \tag{12}$$

In accordance with (10) we can write:

$$\int_R \langle x_{\omega};\wp|x_{\omega};\ \wp\rangle dx_{\omega} =$$
$$= \int_R dx_{\omega} a^*_{\wp}(\omega) \int_R \langle x_{\wp}(\omega);\wp|x'_{\wp}(\omega);\ \wp\rangle dx'_{\wp}(\omega).$$

After the comparison of two expressions we find:

$$a_{\wp}(\omega) = \int_R \langle x_{\wp}(\omega); \wp | x'_{\omega}; \wp \rangle dx'_{\omega}. \quad (13)$$

Note that analogously we receive the discrete variant of (13):

$$a_n = \int_R \langle x_{\wp}(\omega); \wp | x'_{\wp}(\omega); \wp \rangle dx'_{\omega}. \quad (14)$$

To receive the normalization condition of proper information functions, put (14) in (13):

$$a_{\wp}(\omega) =$$
$$= \int_R a'_{\wp}(\omega) \left(\int_R \langle x_{\wp}(\omega); \wp | x'_{\wp}(\omega); \wp \rangle dx'_{\omega}(\omega) \right) dx_{\omega} .$$

This expression must be valid for any values of $a_{\wp}(\omega)$ and, therefore, must be fulfilled identically. From these judgments we can deduce that

$$\int_R \langle x_{\wp}(\omega); \wp | x'_{\wp}(\omega); \wp \rangle dx'_{\wp}(\omega) = \delta \left(x'_{\wp}(\omega) - x_{\wp}(\omega) \right). \quad (15)$$

This formula represents the rule of normalization of proper information functions in the case of continuous spectrum. For the discrete case we have

$$\int_R \langle x_{\omega n}; \wp | x_{\omega m}; \wp \rangle dx_{\omega n} = \delta_{mn}. \quad (16)$$

where δ_{mn} is a Kronecker delta symbol. As $| \, |x_{\omega}; \wp\rangle \, |^2$ represents the probability of x_{ω} in a given interval $(x_{\omega}, x_{\omega} + dx_{\omega})$, the quantity $|a_{\wp}(\omega)|^2$ represents the probability the $\xi_{\wp}(\omega)$ is in the interval $\left(\xi_{\wp}(\omega), \xi_{\wp}(\omega) + d\xi_{\wp}(\omega)\right)$.

There exists a color which has the discrete spectrum in the first part of its values and the continuous spectrum - in another one. In such a case the full system of eigenfunctions is formed from the totality of eigenfunctions of both spectrums. Expansions by such functions have the following form:

$$|x_{\omega}; \wp\rangle = \sum_n a_n |x_{\omega n}; \wp\rangle + \int_R a_{\xi_{\wp}}(\omega) |x_{\omega}; \wp\rangle dx_{\omega}. \quad (17)$$

where the sum is taken by the discrete spectrum and the integration - by the continuous.

The theory of colors in its usual form deals with the characteristics of information vectors and the corresponding operators in Hilbert space: each information state corresponds to the estimation of compatibility function, each color – to operator.

There are several formulations, in the bound of which the information function in phase space (Cartesian product of universal set R on its canonically conjugate R^c) is related to information state or with estimation of experts (observable) colors. Such formalism is based on the Weyl transformation [7]. Information eigenvectors of colors $\wp$ and $\wp^c$ or eigenvectors of operators $\hat{\wp}$ and $\hat{\wp}^c$ satisfy the equations of eigenvalues (6) and (9). The full system of eigenvectors satisfies the completeness condition, which we present here in a form:

$$\int d\xi_{\wp}(\omega) \left| \xi_{\wp}(x_{\omega}); \wp \right\rangle \left\langle \xi_{\wp}; \wp \right| = \hat{E}, \quad (18)$$

and

$$\int d\xi_{\wp^c}(\omega_c) \left| \xi_{\wp^c}(x_{c\omega}); \wp^c \right\rangle \left\langle \xi_{\wp^c}(x_{c\omega}); \wp^c \right| = \hat{E}^c. \quad (19)$$

where $\hat{E}$ $(\hat{E}^c)$ is the unit operator in Hilbert space. Orthogonality conditions are presented by formulae (15) and (16). Commutation relations for canonically conjugate colors will be written in a form:

$$[\hat{\wp}, \hat{\wp}] = 0, \ [\hat{\wp}^c, \hat{\wp}^c] = 0, \ [\hat{\wp}, \hat{\wp}^c] = -2\pi i c \hat{U} \hat{E}, \quad (20)$$

where $\hat{U}$ is a unique Cartesian tensor (three-tensor) of the components δ_{ij} (Kronecker symbol, $i, j = 1,2,3$).

In x_{ω}-representation to eigenvector $|x_{c\omega}; \wp^c\rangle$ corresponds to the information function:

$$\langle x_{\omega}; \wp | x_{c\omega}; \wp^c \rangle = c^{-3/2} e^{\frac{i}{c} x_{\omega} x_{c\omega}}. \quad (21)$$

Using the condition of completeness (18), (19) and introducing the new variables of integration

$$x'_{c\omega} = x_{c\omega} - \frac{1}{2}u, \quad x'_{\omega} = x_{\omega} + \frac{1}{2}v,$$
$$x''_{c\omega} = x_{c\omega} + \frac{1}{2}u, \quad x''_{\omega} = x_{\omega} - \frac{1}{2}v,$$

or $(x'_{\omega}, x''_{\omega}, x'_{c\omega}, x''_{c\omega}) \to (x_{\omega}, x_{c\omega}, u, v)$ with Jacobian equal to one, the following equality for an arbitrary operator $\hat{A}$ can be written [7]:

$$\hat{A} = (2\pi c)^{-3} \int_R dx_{c\omega} dx_{\omega} W(x_{\omega}; x_{c\omega}) \hat{\Delta}(x_{\omega}; x_{c\omega}), \quad (22)$$

where the function

$$W(x_{\omega}; x_{c\omega}) = \int_R du \cdot e^{\frac{i}{2\pi c} x_{\omega} u} \left\langle x_{c\omega} + \tfrac{1}{2}u \middle| x_{c\omega} - \tfrac{1}{2}u \right\rangle \quad (23)$$

is called the Weyl transformation of operator $\hat{A}$ by the operators $\hat{\wp}$ and $\hat{\wp}^c$, and

$$\hat{\Delta}(x_{\omega}; x_{c\omega}) = \int_R dv \cdot e^{-\frac{i}{2\pi c} x_{c\omega} x_{\omega}} \left| x_{\omega} + \tfrac{1}{2}v; \wp \right\rangle \left\langle x_{\omega} - \tfrac{1}{2}v; \wp \right|. \quad (24)$$

This Hermitian operator doesn't depend on $\hat{A}$.

Thus, each operator (q-number) can be corresponded to the c-number. If $\hat{A}$ is a Hermitian operator, then $W(x_{\omega}, x_{c\omega})$ is a real function.

If as an $\hat{A}$ to take the operator of characteristic function of color $\wp \times \wp^c$ [4, 5],

$$\hat{M}(\alpha_1, \alpha_2) = exp[i(\alpha_1 \hat{\wp} + \alpha_2 \hat{\wp}^c)],$$

then the formula (23) takes the following form:

$$W_{\wp \times \wp^c}(x_{\omega}, x_{c\omega}) =$$
$$= \frac{1}{2\pi} \int_R \langle x_{\omega} - \pi c v | e^{-i v x_{c\omega}} | x_{\omega} + \pi c v \rangle dv. \quad (25)$$

The formula (25) for the density $W_{\wp \times \wp^c}(x_{\omega}, x_{c\omega})$ allows to calculate the conditional computable values of color.

Let's introduce the conditional characteristic function of color:

$$M(\alpha | x_{\omega}) = \frac{1}{\rho_{\wp}(x_{\omega})} \int_R W_{\wp \times \wp^c}(x_{\omega}, x_{c\omega}) e^{i\alpha x_{c\omega}} dx_{c\omega} =$$
$$= \frac{1}{\rho_{\wp}(x_{\omega})} \int_R dx_{c\omega} e^{i\alpha x_{c\omega}} \frac{1}{2\pi} \int_R \left\langle x_{\omega} - \frac{c\alpha'}{2}; \wp \middle| x_{\omega} + \frac{c\alpha'}{2}; \wp \right\rangle \times$$
$$\times e^{-i\alpha x_{c\omega}} d\alpha' = \frac{1}{\rho_{\wp}(x_{\omega})} \int_R d\alpha' \left\langle x_{\omega} - \frac{c\alpha'}{2}; \wp \middle| x_{\omega} + \frac{c\alpha'}{2}; \wp \right\rangle \times$$
$$\frac{1}{2\pi} \int_R e^{i(\alpha_c - \alpha'_c) x_{c\omega}} dx_{c\omega} = \frac{1}{\rho_{\wp}(x_{\omega})} \left\langle x_{\omega} - \frac{c\alpha}{2}; \wp \middle| x_{\omega} + \frac{c\alpha}{2}; \wp \right\rangle. \quad (26)$$

Supposing

$$|x_{\omega}; \wp\rangle = \rho_{\wp}^{1/2}(x_{\omega}) e^{\frac{i}{c} S(x_{\omega})}, \quad (27)$$

one can write the logarithm $M(\alpha | x_{\omega})$, or "cumulant" function in a form:

$$K(\alpha | x_{\omega}) = \ln M(\alpha | x_{\omega}) = \frac{1}{2} ln\rho_{\wp}\left(x_{\omega} + \frac{c\alpha}{2}\right) + \frac{1}{2} ln\rho_{\wp}\left(x_{\omega} - \frac{c\alpha}{2}\right) - ln\rho_{\wp}(x_{\omega}) + \frac{i}{c}\left(S\left(x_{\omega} + \frac{c\alpha}{2}\right) - S\left(x_{\omega} - \frac{c\alpha}{2}\right)\right). \quad (28)$$

Thus, for cumulants $\bar{\chi}_n(x_\omega)$ of the given distribution (coefficients attached to $\frac{1}{n!}(\alpha c)^n$ in Taylor series of $K(\alpha|x_\omega)$ we receive the following expression:

$$\bar{\chi}_{2n+1}(x_\omega) = \left(\frac{c}{2i}\right)^{2n} \frac{d^{2n+1}}{dx_\omega^{2n+1}} S(x),\ n = 0,1,\cdots \quad (29)$$

$$\bar{\chi}_{2n}(x_\omega) = \left(\frac{c}{2i}\right)^{2n} \frac{d^{2n}}{dx_\omega^{2n}} ln\rho_\wp(x_\omega),\ n = 0,1,\cdots \quad (30)$$

The quantities $\bar{\chi}_n$ are simply connected with calculated values $(x"_{c\omega})^*_{x_\omega}$. Particularly, for $n = 1$ we have:

$$\bar{\chi}_1(x_\omega) = (x_{c\omega})^*_\omega = \frac{dS}{dx_\omega}. \quad (31)$$

This permits to interpret the argument of complex function $|x_\omega;\ \wp\rangle,\ S(x_\omega)$ as potential of conditional calculated values (conditional mathematical expectation).
The conditional dispersion of color $\wp^c$ is:

$$\bar{\chi}_2(x_\omega) = \sigma^2_{\wp^c} = (x^2_{c\omega})^* - (x^*_\omega)^2 = \frac{c^2}{4}\frac{d^2}{dx^2_\omega} ln\rho_\wp(x_\omega). \quad (32)$$

Asymmetry of distribution is determined only by its odd semi-invariants. Thus, asymmetry of conditional distribution of numerical values of canonically conjugate color depends only on $S(x_\omega)$.

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