Semi-Markov Queuing System with Bifurcation of Arrivals for Network Maintenance Problem

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ABSTRACT
In the present paper, a multi-unit redundant system with unreliable, repairable units is considered. Two types of maintenance operations - the replacement of the failed main unit by the redundant one and the repair of the failed unit - are performed. The case of the system with one replacement server with arbitrary replacement time distribution function and repair server with an exponential distribution of repair time is considered. For this system mixed-type semi-Markov queuing model with the bifurcation of arrivals is constructed. It represents a non-classical boundary value problem of mathematical physics with non-local boundary conditions.

Keywords
Replacement, repair, semi-Markov queuing model, bifurcation.

1. INTRODUCTION
The research topic belongs to the Mathematical Theory of Reliability (MTR), which applies the methods and models of Queuing Theory (QT) while investigating dependability and performability analysis of complex systems. We consider a multi-unit redundant system with repairable units, such as info-communication, computer and transportation networks, power and defense systems, etc. The problem is formulated in terms of the practical problems of Queuing Theory, for the solution of which, the necessity of our research has arisen. It's necessary to underline the significance of the research topic as well as its place within the network maintenance problem [1-6]. The open and mix type queuing models for dependability and performability analysis of territorially distributed networks is quite important. The fact is that for a long period of time it was believed that in MTR and QT, in problems of reliability and maintenance of redundant complex systems only finite-source (closed) queuing models are applicable. However, this idea is valid for classical machine maintenance problem. For modern network maintenance problem open queuing models or mixed type models are mainly applied. This is convincingly verified by experts from Georgian Technical University in their publications for last years [7-12].

The research subject consists of m ($m = \infty$) identical main (operative) and n redundant units. For the normal operation of the system it's supposed that all m units be operative in the set of main units. But if their number is less, the system continuous functioning with reduced economical effectiveness. The total failure rate of the set of all main units is α and the failure rate of individual redundant units is β.

For illustration we can consider a specific example – the number of Radio Base Stations (RBS) in modern mobile communication networks may be hundreds, thousands and more. That means that in mathematical models (as it is accepted in QT) the set of RBS can be consider as an infinite ($m = \infty$) source of failures. Due to the same factor we can consider the total failure rate to be constant. Consequently, we will have a Poisson stream of requests to maintenance facilities. As it is known, this is very important for construction and investigation of suitable mathematical models.

The failed main unit must be replaced by operative redundant one. Thus, if at failure moment there is a free redundant unit in the system, its replacement operation will happen. The failed units, both main and redundant ones, must be repaired. The repaired unit is supposed to be identical to the new one.

In general statement of the problem, there are k and l maintenance servers for replacement and repair, respectively. Replacement and repair times are random variables with H and G distribution functions. In case when servers are busy, request queues for replacement or repair are formed. The service discipline is FCFS (First Come, First Served). This is a mixed type queuing system (queuing system with infinite and finite sources simultaneously) with two types of services – replacement and repair. The request for replacement arises when the main unit fails. The same event, as well as the failure of a redundant unit, generates the request for the repair (see Fig. 1).

Thus, we have a queuing system, which is open with respect to the main units' failures stream (source of requests is infinite) and, at the same time, it is a closed queuing system with respect to the redundant units' failures stream (source of requests is finite). The diagram in Fig. 1 explains queuing models of network maintenance problem [7].

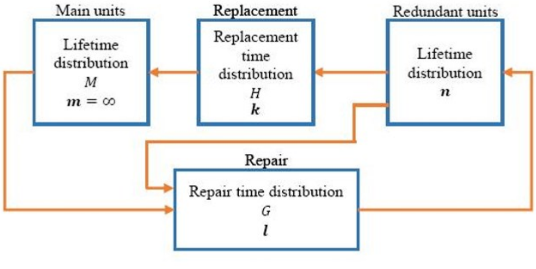

Fig. 1. Diagram illustrating the network maintenance problem.

From the above description it is obvious that the failure of main units generates two requests for service: 1) the replacement of failed unit by redundant one; 2) the repair of failed unit. That is called the bifurcation of arrivals (failures) [7]. Therefore, both of maintenance operations should be performed in parallel mode.

The Markov models to similar problem were discussed in earlier publications [7-11] of the authors of the present paper. In these papers we discussed the exponential models of our research subject with bifurcation of arrivals. For them the suitable mathematical models as infinite systems of ordinary differential equations were constructed. These equations in steady state conditions were reduced to the infinite system of linear algebraic equations. The main difficulty in building and analyzing the mathematical model of the discussed system was the bifurcation of arrivals. Overcoming this difficulty was the main challenge to the model obtained.

The semi-Markov queuing model considering the bifurcation of failures was considered in our paper [12] in case of one repair server and one replacement server (k = l = 1). Replacement time had an exponential distribution while repair time distribution function is arbitrary. Unlike exponential systems we obtained there much more complex mathematical description.

2. SEMI-MARKOV MODEL

Assume we have one replacement server, with an arbitrary replacement time distribution function $H(t)$, and l repair server. The repair time has an exponential distribution with parameter μ. The failed units, both main and redundant ones, must be repaired. The repaired unit is supposed to be identical to the new one.

To describe the system state at the time instant t, the following stochastic processes are introduced:

$i(t)$ - is the number of units missing in the group of main units;

$j(t)$ - is the number of failed units in the system;

$\xi(t)$ - is the time interval length from the beginning of the repair operation to the time instant t.

Denote by $\lambda(u)=\dfrac{H'(u)}{1-H(u)}$ repair rate.

To describe the system state, the following probabilistic characteristics are defined:

$$P(j,t)=P\{i(t)=0,\quad j(t)=j\},\ j=\overline{0,n},$$
$$R(i,t)=P\{i(t)=i,\quad j(t)=n+i\},\ i=\overline{1,m},$$
$$q(i,j,t,u)=\lim\left(\frac{1}{h}P\{i(t)=i,\ j(t)=j-1,\ u<\xi(t)<u+h\}\right),$$
$$i=\overline{1,m},\ j=\overline{1,n+i}.$$

Suppose,

A) $P(j,t)$ and $R(i,t)$ functions have continued derivatives when $t>0$;

B) $q(i,j,t,u)$ functions have continued partial derivatives when $t>0$ and $u\geq 0$.

Assume that $P(0,0)=1$, then $P(j,0)=0,\quad j>0$, $R(i,0)=0,\quad i>0$ and $q(i,j,0,u)=0$ for any i and j.

We will consider the case $l\leq n.$ According to the usual probabilistic considerations, the following theorems are proved.

Theorem 1. Suppose, the conditions A) and B) are valid. Then, the functions $P(j,t)$ and $R(i,t)$ satisfy the following system of integro-differential equations:

$$\frac{dP(0,t)}{dt}=-(\alpha+n\beta)P(0,t)+\mu P(1,t)+\int_0^t q(1,1,t,u)\lambda(u)du;$$
$$\frac{dP(j,t)}{dt}=-(\alpha+(n-j)\beta+j\mu)P(j,t)+(n-(j-1))\times$$
$$\times\beta P(j-1,t)+(j+1)\mu P(j+1,t)+\int_0^t q(1,j+1,t,u)\lambda(u)du,$$
$$1\leq j<l; \qquad (1)$$
$$\frac{dP(j,t)}{dt}=-(\alpha+(n-j)\beta+l\mu)P(j,t)+(n-(j-1))\times$$
$$\times\beta P(j-1,t)+l\mu P(j+1,t)+\int_0^t q(1,j+1,t,u)\lambda(u)du,$$
$$l<j<n;$$
$$\frac{dP(n,t)}{dt}=-(\alpha+\mu)P(j,t)+\beta P(j-1,t)+$$
$$+\int_0^t q(1,n+1,t,u)\lambda(u)du;$$
$$\frac{dR(i,t)}{dt}=-(\alpha+l\mu)R(i,t)+\alpha R(i-1,t)+$$
$$+\int_0^t q(i+1,n+i+1,t,u)\lambda(u)du,\quad i\geq 1;$$

Note that $R(0,t)=P(n,t)$.

Theorem 2. Suppose, the condition B) is valid. The functions $q(i,j,t,u)$ satisfy the following infinite system of partial differential equations:

$$\frac{\partial q(1,1,t,u)}{\partial t}+\frac{\partial q(1,1,t,u)}{\partial u}=-(\alpha+n\beta+\lambda(u))\times$$
$$\times q(1,1,t,u)+\mu q(1,2,t,u);$$
$$\frac{\partial q(1,j,t,u)}{\partial t}+\frac{\partial q(1,j,t,u)}{\partial u}=-(\alpha+(n+1-j)\beta+j\mu+\lambda(u))\times$$
$$\times q(1,j,t,u)+(n+1-(j-1))\beta q(1,j-1,t,u)+$$
$$+j\mu q(1j+,1,t,u),\quad 1<j\leq l; \qquad (2)$$
$$\frac{\partial q(1,j,t,u)}{\partial t}+\frac{\partial q(1,j,t,u)}{\partial u}=-(\alpha+(n+1-j)\beta+l\mu+\lambda(u))\times$$
$$\times q(1,j,t,u)+(n+1-(j-1))\beta q(1,j-1,t,u)+$$
$$l\mu q(1j+,1,t,u),\qquad l<j\leq n;$$
$$\frac{\partial q(1,n+1,t,u)}{\partial t}+\frac{\partial q(1,n+1,t,u)}{\partial u}=-(\alpha+l\mu+\lambda(u))\times$$
$$\times q(1,n+1,,t,u)+\beta q(1,n,t,u);$$
$$\frac{\partial q(i,1,t,u)}{\partial t}+\frac{\partial q(i,1,t,u)}{\partial u}=-(\alpha+(n+i-1)\beta+\lambda(u))\times$$
$$\times q(i,1,t,u)+\mu q(i,2,t,u),\qquad i>1;$$

$$\frac{\partial q(i,j,t,u)}{\partial t}+\frac{\partial q(i,j,t,u)}{\partial u}=-(\alpha+(n+i-j)\beta+j\mu+\lambda(u))\times$$
$$\times q(i,j,t,u)+\alpha q(i-1,j-1,t,u)+(n+i-(j-1))\beta\times$$
$$\times q(i,j-1,t,u)+j\mu q(i,j+1,t,u),\ i>1,\quad 1<j\le l;$$
$$\frac{\partial q(i,j,t,u)}{\partial t}+\frac{\partial q(i,j,t,u)}{\partial u}=-(\alpha+(n+i-j)\beta+l\mu+\lambda(u))\times$$
$$\times q(i,j,t,u)+\alpha q(i-1,j-1,t,u)+(n+i-(j-1))\times$$
$$\times \beta q(i,j-1,t,u)+l\mu q(i,j+1,t,u),\ i>1,\quad l<j<n+i;$$
$$\frac{\partial q(i,n+i,t,u)}{\partial t}+\frac{\partial q(i,n+i,t,u)}{\partial u}=-(\alpha+l\mu+\lambda(u))\times$$
$$\times q(i,n+i,t,u)+\alpha q(i-1,n+i-1,t,u)+\beta q(i,j-1,t,u),$$
$$i>1.$$

Theorem 3. The boundary conditions for the functions $q(i,j,t,0)$ are given using the following infinite system of recursive relations:

$$q(1,j,t,0)=\alpha P(j-2,t)+$$
$$+\int_0^t q(2,j+1,t,u)\lambda(u)du+0(h),\quad 1<j\le n;$$
$$q(i,j,t,0)=\int_0^t q(i+1,j+1,t,u)\lambda(u)du,$$
$$i>2,\quad 1<j<n+i; \tag{3}$$
$$q(i,n+i,t,0)=l\mu R(1,t),\quad i>2\ .$$

The systems of equations (1), (2), (3) represent non-classical boundary value problem of mathematical physics with non-local boundary conditions (3).

3. CONCLUSION

The bifurcation of arrivals means that the failure of main units generates two requests for service: the replacement of failed unit by redundant one and the repair of failed unit. Therefore, both maintenance operations should be performed in parallel mode. In the considered case, unlike exponential systems, we obtained a much more complex mathematical model, namely, described by integro-differential equations, infinite system of partial differential equations and the infinite system of recursive functional equations. Overall, they formed the non-classical boundary value problem of mathematical physics. At present the problem is still being investigated. Investigation of this task (the existence and uniqueness of the solution) is associated with significant difficulties. Direction for future research can be the investigation of system's behavior for different distribution functions of repair time as well.

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