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On d-Panconnected Tournaments with Large Semi-Degrees

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***Abstract*—We prove the following new results:**

(a) *Let T be a tournament of order $2n+1 \geq 11$ and S a subset of $V(T)$. Suppose that $|S| \leq \frac{1}{2}(n-2)$ and x, y are distinct vertices in $V(T) \setminus S$. If the subtournament $T-S$ contains an (x,y)-path of length r, where $3 \leq r \leq 2n-|S|-1$, then $T-S$ also contains an (x,y)-path of length $r+1$.*

(b) *Let T be an m-irregular tournament of order p, i. e, $|d^+(x)-d^-(x)| \leq m$ for every vertex of T. If $m \leq \frac{1}{3}(p-5)$ (respectively, $m \leq \frac{1}{5}(p-3)$), then for every pair of vertices x and y, T has an (x,y)-path of any length k, $4 \leq k \leq p-1$ (respectively, $3 \leq k \leq p-1$ or T belongs to a family $\mathcal{G}$ of tournaments, which is defined in the paper). In other words, (b) means that if the semidegrees of every vertex of a tournament T of order p are between $\frac{1}{3}(p+1)$ and $\frac{2}{3}(p-2)$ (respectively, between $\frac{1}{5}(2p-1)$ and $\frac{1}{5}(3p-4)$), then the claims in (b) are true.*

Our results improve in a sense related results Alspach (1967), Jacobsen (1972), Alspach et al. (1974), Thomassen (1978) and Darbinyan (1977, 1978, 1979) and are sharp in some sence.

***Keywords*— tournaments; arc pancyclicity; irregularity; paths; panconnected tournaments; outdegree; indegree.**

I. Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. We use standard notation and terminology, cf. [1] and [2]. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices. A subdigraph of D induced by a subset $A \subseteq V(D)$ is denoted by $D\langle A\rangle$. If $X \subseteq V(D)$, then $D-X$ is the subdigraph induced by $V(D)\setminus X$, i.e., $D-X = D\langle V(D)\setminus X\rangle$. Every cycle and path are assumed to be simple and directed. Let m and n, $m \leq n$, be two integers. By $[m,n]$ we denote the set $\{m, m+1, \ldots, n\}$.

A digraph D of order p is *arc pancyclic* (respectively, d-*arc pancyclic*, where $d \in [3,p]$) if D has a k-cycle containing an arc uv for every arc $uv \in A(D)$ and every $k \in [3,p]$ (respectively, $k \in [d,n]$). We say that a digraph D of order p is *strongly panconnected* (respectively, d-*strongly panconnected, where* $d \in [3,p-1]$ if there is an (x,y)- and a (y,x)-path in D, both of length k, for any two vertices x, y of D and each $k \in [3,p-1]$ (respectively, $k \in [d,p-1]$).

An oriented graph is a digraph with no cycle of length two. A tournament is an oriented graph, where every pair of distinct vertices are adjacent. The irregularity $i(T)$ of a tournament T is the maximum $|d^+(x)-d^-(x)|$ over all vertices x of T. If $i(T)=0$, then T is regular and if $i(T)=1$, then T is almost regular. Observe that every vertex of a tournament T of order p has outdegree between $0.5(p-1+i(T))$ and $0.5(p-1+i(T))$.

There are a number of conditions that guarantee that a tournament is arc pancyclic or strongly panconnected (see, e.g., [1]- [18]). In particular, Alspach [3] proved that every regular tournament is 3-arc pancyclic. Jacobsen [12] proved that every almost regular tournaments of order $p \geq 8$ is 4-arc pancyclic. Alspach et al. [4] proved that every regular tournament of order $p \geq 7$ is 3-strongly panconnected. Darbinyan [9] proved that every almost regular tournament of order $p \geq 10$ is 3-strongly panconnected.

Thomassen [17] generalized these results as follows:

Theorem 1: Let T be an m-irregular tournament of order $p \geq 4$. If $m \leq \frac{1}{5}(p-9)$, then T is 3-strong panconnected. If $m \leq \frac{1}{5}(p-3)$, then T is 4-arc pancyclic.

In [8] and [10], Darbinyan obtained the following:

Theorem 2: Let T be a regular tournament of order $2n+1$ and let $S \subset V(T)$.

(i) [8]. If $2 \leq |S| \leq \lfloor \frac{1}{3}(n-2) \rfloor$, then $T-S$ is 3-strongly panconnected.

((ii) [10]). Let $|S| \leq \lfloor \frac{1}{2}(n-3) \rfloor$ and $x, y \in V(T) \setminus S$ be two distinct vertices. If $T-S$ contains an (x,y)-path of length r, where $r \in [3, 2n-|S|-1]$, then $T-S$ also contains an (x,y)-path of length $r+1$.

We will use the following result of Moon [14].

Theorem 3: Let H be an m-irregular tournament of order $p \geq 2$. Then there is a regular tournament of order $p+m$ such that T contains H as a subtournament.

From Theorems 2(ii) and 3 it is not difficult to obtain the following:

Corollary: ([10]). *Let T be an m-irregular tournament of order $p \geq 7$, where $m \leq \frac{1}{3}(p-7)$, and x, y are two distinct vertices in T. If T contains an (x,y)-path of length r, where $r \in [3, p-2]$, then T also contains an (x,y)-path of length $r+1$.*

In this paper, we prove the following theorems, which improve in a sense the above-mentioned results of Alspach, Jacobsen, by Alspach et al., Thomassen and Darbinyan. The following theorem is our main results.

Theorem 4: Let T be a regular tournament of order $2n+1 \geq 11$ and let S be a subset in $V(T)$. Suppose that $|S| \leq \frac{1}{2}(n-2)$ and x, y are two distinct vertices in $V(T)\setminus S$. If $T-S$ contains an (x,y)-path of length r, where $r \in [3, 2n-|S|-1]$, then $T-S$ also contains an (x,y)-path of length $r+1$.

The following remark shows that Theorem 1.5 is sharp in a sense.

Remark 1: If in Theorem 1.5 we replace $\frac{1}{2}(n-2)$ with $\frac{1}{2}(n-1)$ (respectively, $\frac{1}{2}(n)$), then there is a regular tournament of order $2n+1 = 11$ (respectively, $2n+1 = 13$) which contains a subset $S \subset V(T)$ with $|S| = \frac{1}{2}(n-1)$ (respectively, $|S| = \frac{1}{2}(n)$) and two distinct vertices $x, y \in V(T) \setminus S$ such that $T-S$ contains an (x,y)-path of length 3, but $T-S$ contains no (x,y)-path of length 4.

Here we define only a regular tournament T of order $2n+1 = 11$ as follows:

$V(T) = \{x_0, x_1, x_2, x_3\} \cup A \cup S$ such that $|A| = 5$ and $|S| = 2$. Let $A = \{u_1, u_2, u_3, u_4, z\}$ and $S = \{v_1, v_2\}$. Moreover, T satisfies the following properties: $N^-(x_0) = A$, $N^-(x_1) = \{x_0, z, u_3, u_4, v_2\}$, $N^-(x_2) = \{x_0, x_1, z, u_3, u_4\}$, $N^-(x_3) = \{x_0, x_1, x_2\} \cup S$, $N^-(z) = \{x_3, u_1, u_2, u_3, u_4\}$, $N^-(u_1) = \{x_1, x_2, x_3, u_4, v_2\}$, $N^-(u_2) = \{x_1, x_2, x_3, u_1, v_1\}$, $N^-(u_3) = \{x_3, u_1, u_2, v_1, v_2\}$, $N^-(u_4) = \{x_3, u_2, u_3, v_1, v_2\}$, $N^-(v_1) = \{x_0, x_1, x_2, z, u_1\}$ and $N^-(v_2) = \{x_0, x_2, z, v_1, u_2\}$.

It is not difficult to check that T contains an (x_0, x_3)-path of length 3, but contains no (x_0, x_3)-path of length 4.

Using Theorems 3, 4 and Lemma 2 in Section 3, we obtain the following:.

Theorem 5: Let T be an m-irregular tournament of order p such that $p+m \geq 11$. If $m \leq \frac{1}{3}(p-5)$ (respectively, $m \leq \frac{1}{5}(p-3)$), then T is 4-strongly panconnected (respectively, T is 3-strongly pannconnected or T belongs to the family $\mathcal{G}$ of tournaments defined in Section 3).

II. Further terminology and notation

Let x, y be distinct vertices in D. The arc of a digraph D directed from x to y is denoted by xy (we say that x dominates y). For disjoint subsets B and C of $V(D)$, let $A(B \to C) = \{xy \in A(D) \,|\, x \in A, y \in B\}$. We write $B \to C$ if every vertex of B dominates every vertex of C. If $A \subseteq V(D) \setminus (B \cup C$, then $A \to B \to C$ is a shortcut for $A \to B$ and $B \to C$. If $x \in V(D)$ and $A = \{x\}$, we write x instead of $\{x\}$.

The *out-neighborhood* (respectively, *in-neighborhood*) of a vertex x is the set $N^+(x) = \{y \in V(D) \,|\, xy \in A(D)\}$ (respectively,$N^-(x) = \{y \in V(D) \,|\, yx \in A(D)\}$). Similarly, if $B \subseteq V(D)$, then $N^+(x, B) = \{y \in B \,|\, xy \in A(D)\}$ and $N^-(x, B) = \{y \in B \,|\, yx \in A(D)\}$. Note that for a vertex x, we have $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$. Similarly, let $d^+(x, B) = |N^+(x, B)|$ and $d^-(x, B) = |N^-(x, B)|$.

The *path* (respectively, the *cycle*) in D consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). The *length* of a cycle or path is the number of its arcs. A *k-cycle* is a cycle of length k. We say that $x_1x_2 \ldots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-path. In this paper, we will use *the principle of digraph duality*, which is as follows: Let D be a digraph and $\overleftarrow{D}$ be the converse digraph of D. Then D contains a subdigraph H if and only if $\overleftarrow{D}$ contains the converse of $\overleftarrow{H}$.

III. Sketch of the Proof of the main result

The following lemma states well-known and simple claims, which are the basis of our results and other theorems on directed cycles and paths in tournaments. The claims will be used extensively in the proof of our result.

Lemma 1: Let T be a regular tournament of order $p \geq 2$. Then the following statements are true.

(i) The tournament T contains two distinct vertices x and y (respectively, u and v) such that $d^-(x) \leq 0.5(p-1)$ and $d^-(y) \geq 0.5(p-1)$ (respectively, $d^+(u) \leq 0.5(p-1)$ and $d^+(v) \geq 0.5(p-1)$).

(ii) If T is regular, then $p = 2n+1$ and for any vertex $x \in V(T)$, $d^-(x) = d^+(x) = n$.

(iii) If T is not regular, then T contains two distinct vertices x and y (respectively, u and v) such that $d^-(x) \leq 0.5(p-2)$ and $d^-(y) \geq 0.5p$ (respectively, $d^+(u) \leq 0.5(p-2)$ and $d^+(v) \geq 0.5p$.

(iv) If T is almost regular, then $p = 2n$ and n vertices have indegrees equal to n and the other n vertices have outdegrees equal to n.

(v) Let T be a not regular tournament of order $p \geq 2$. If for all $v \in V(T)$, $d^-(v) < (p+1)/2$ (or $d^+(v) < (p+1)/2$), then T is almost regular.

(vi) Let T be a tournament of order $p \geq 2$. If for all $v \in V(T)$, $d^-(v) < p/2$ (or $d^+(v) < p/2$), then T is regular.

To formulate Lemma 2, we need the following definition.

Definition 1: By $\mathcal{G}$ we denote the set of tournaments, each of which has order $6k+3 \geq 9$ and vertex set $\{x, y, z\} \cup A \cup B \cup C \cup S$ with the properties $|A| = |C| = 2k-1$, $|B| = k+2$, $|S| = k$, the subtournaments induced by the subsets A, C, $\{z\} \cup B \cup S$ are regular, $A \to B \cup S \to C$, $x \to \{y, z\} \cup B \cup C$, $x \to \{y, z\} \cup A \cup S$, $\{x, z\} \cup C \cup S \to y$, $y \to A \cup B$ and $B \cup C \to x$.

Let $G \in \mathcal{G}$. Then $G - S$ has no (x, y)-path of length 3.

Remark 2: It is interesting that Thomassen cite[17]used tournaments of the form $G - S$ where $G \in \mathcal{G}$ to show that there are many tournaments of order p with irregularity equal to $\frac{1}{5}(p-3)$, which are not 3-strongly panconnected.

Lemma 2: Let T be a regular tournament of order $2n+1$. Suppose that $S \subseteq V(T)$ and x, y are two distinct vertices in $V(T) \setminus S$. Then the following hold:

(i) If $n \geq 3$ and $k = |S| \leq \frac{1}{3}(n-1)$, then $T-S$ contains an (x, y)-path of length 3, unless T is isomorphic to G.

(ii) If $n \geq 5$ and $k = |S| \leq \frac{1}{2}n$ and $T-S$ contains no (x, y)-path of length 3, then $T-S$ contains an (x, y)-path of length 4.

In Lemmas 3-4, we suppose that $P := x_0x_1 \ldots x_r$ is an (x_0, x_r)-path of length r in a tournament T and z is a vertex in $V(T) \setminus V(P)$ such that $\{x_{\alpha+1}, x_{\alpha+2}, \ldots, x_r\} \to z \to$

$\{x_0, x_1, \ldots, x_\alpha\}$, where $\alpha \in [2, r-3]$. Moreover, by Q is denoted any (x_0, x_r)-path of length $r+1$ with vertex set $\{z\} \cup V(P)$, and is assumed that T contains no such Q paths:

Lemma 3: Suppose that $x_s x_t \in A(T)$ with $s \in [2, \alpha - 1]$ and $t \in [\alpha + 3, r]$. Then
$A(\{x_0, x_1, \ldots, x_{s-2}\} \rightarrow \{x_{\alpha+2}, x_{\alpha+3}, \ldots, x_{t-1}\}) = \emptyset$, when $s \geq 3$ and $A(x_{s-1} \rightarrow \{x_{\alpha+2}, x_{\alpha+3}, \ldots, x_{t-1}\}) = \emptyset$, when $t - s \neq 5$.

Lemma 4: Suppose that $x_s x_t \in A(T)$ with $s \in [\alpha, r-2]$, $t \in [s+2, r]$. If $k = \lfloor \frac{1}{2}(t-s) \rfloor$, then
$A(\{x_1, x_2, \ldots, x_{\alpha-1}\} \rightarrow \{x_{s+1}, x_{s+2}, \ldots, x_{s+k}\}) = \emptyset$. Let Q denotes any (x_0, x_r)-path of length $r+1$ in $T - S$. By contradiction, suppose that $T - S$ has no such Q path.

Put $A = V(T) \setminus (V(P) \cup S)$. First, we prove a series of claims. In particular, they include the following three claims.

Claim 1: $N^-(x_r, A) = N^+(x_0, A) = \emptyset$, i.e., $d^-(x_r, A) = d^+(x_0, A) = 0$.

Claim 2: $|A| \leq k+1$.

Claim 3: $N^+(x_2, A) = N^-(x_{r-2}, A) = \emptyset$.

Then we divide the proof into two cases to consider.

Case I: $A(\{x_1, x_2, \ldots, x_{\alpha-1}\} \rightarrow x_r) = \emptyset$.

Case II: $A(\{x_1, x_2, \ldots, x_{\alpha-1}\} \rightarrow x_r) \neq \emptyset$.

For Case I, we first need to show a number of claims, which we are not present here. Case II follows from Case I using Lemmas 3, 4 and the principle of digraph duality.

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