Complete Caps in Projective Geometry PG$(n, 3)$

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***Abstract*—We consider the problem of finding the sizes of the largest and the least caps in projective geometry PG$(\mathbf{n}, \mathbf{3})$ over the field $\mathbf{F_3} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. A cap is a set of points, no three of which are collinear. We give two new recurrence constructions for complete caps in projective geometry PG$(\mathbf{n}, \mathbf{3})$. Notice that the constructed caps for some n′s have maximal possible sizes.**

***Keywords*—Affine space, projective space, points, caps, complete caps.**

I. INTRODUCTION

In this paper, we consider a variant of the packing problem for the n-dimensional projective geometry PG(n, q) over a finite field F_q with q elements. The packing problem is to find the maximum cardinality of a set points with property that no k points from this set are linearly dependent. When $k = 3$ such sets are called caps. A cap is called complete when it cannot be extended to a larger one. The main problem in the theory of caps is to find the minimal and maximal sizes of complete caps in projective geometry PG(n, q) and/or in affine geometry AG$(n, 3)$. Finding the exact value for minimal and maximal cardinality of caps in projective geometry PG (n, q) or in affine geometry AG$(n, 3)$, in the general case, seems to be a very hard problem. There are some well-known constructions (doubling, product and recursive), which allow to create large high-dimensional caps based on large low-dimensional caps. Note that the problem of determining the minimum size of a complete cap in a given space is of particular interest in the Coding theory. If we write the points of the cap as columns of a matrix, we obtain a matrix in which every three columns are linearly independent, hence the generator matrix of a linear orthogonal array of strength three. This matrix is a check matrix of a linear code with minimum distance greater than three. Let us denote the size of the largest caps in AG(n, q) and PG(n, q) by $s_{n,q}$ and by $s'_{n,q}$, respectively. Presently, only the following exact values are known: $s_{n,2} = s'_{n,2} = 2^n$, $s_{2,q} = s'_{2,q} = q + 1$ if q is odd, $s_{2,q} = s'_{2,q} = q + 2$ if q is even, and $s'_{3,q} = q^2 + 1$, $s_{3,q} = q^2$ [1,2]. Apart from these general results, the precise values are known in the following cases: $s_{4,3} = s'_{4,3} = 20$ [3], $s'_{5,3} = 56$ [4], $s_{5,3} = 45$ [5], $s'_{4,4} = 41$[6], $s_{6,3} = 112$ [7]. In the other cases, lower and upper bounds on the sizes of caps in AG(n, q) and PG(n, q) are known [8]. The trivial lower bound for the size of the smallest complete cap in AG(n, q) is $\sqrt{2}q^{\frac{n-1}{2}}$ [9]. For q even there exist complete caps in AG(n, q) with less than $q^{\frac{n}{2}}$ points. But for q odd complete caps in AG(n, q) with less than $q^{\frac{n}{2}}$ points are known to exist only for $n \equiv 0(mod\ 4)$, $n \equiv 2(mod\ 4)$ and for small values of n and q [10]. In this paper, we give two new recurrence constructions for complete caps in projective geometry PG$(n, 3)$, which implies some well-known results.

II. MAIN RESULTS

It is easy to see that if $\boldsymbol{S}$ is a cap in AG$(n, 3)$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \not\equiv \mathbf{0}\ (mod\ 3)$ for every triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \boldsymbol{S}$. Let's denote by $B_n = \{(\alpha_1, \cdots, \alpha_n)\ /\alpha_i = 1, 2\}$, by $B'_n = \{(\alpha_1, \cdots, \alpha_n)\ /\ \alpha_1 = 1, \alpha_i = 1, 2;\ 2 \le i \le n\}$ and by P_n the set of points of AG$(n, 3)$ satisfying the following two conditions:

i) for any two distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta} \in P_n$, there exists $i (1 \le i \le n)$so that $\alpha_i = \beta_i = 0$,

ii) for any triple of distinct points $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in P_n$, $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \not\equiv \mathbf{0}(mod\ 3)$.

We say P_n to be complete when it cannot be extended to a larger one. We will define the concatenation of the sets points in the following way. Let $A \subset AG(n, 3)$ and $B \subset AG(m, 3)$. We form a new set $AB \subset AG(n + m, 3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n,\ \alpha_{n+1}, \cdots, \alpha_{n+m})$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_n) \in A$ and $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, \cdots, \alpha_{n+m}) \in B$. In a similar way, one can define the concatenation of the points for any number of sets.

Claim 1. Note that if $x, y, z \in F_3$, then $x + y + z \equiv 0\ (mod\ 3)$ if and only if $x = y = z$ or they are pairwise distinct.

Theorem 1[11]**.** The following recurrence relation $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$, with initial sets $P_1 = \{(0)\}$, $P_2 = \{(0,1), (0,2)\}$ and $n = \sum_{j=1}^{3} n_j$, yields a complete P_n set.

Proof. We use induction on n. It is obvious that $P_1 = \{(0)\}$ and $P_2 = \{(0,1), (0,2)\}$ are complete and they satisfy the conditions i) and ii). It is not difficult to check that

$P_3 = \{(0,0,1),(0,0,2),(0,1,0),(0,2,0),(1,0,0),(2,0,0)\}$ is also complete and satisfies the conditions i) and ii). Assume that the sets P_{n_1}, P_{n_2} and P_{n_3} are complete and satisfy the conditions i) and ii). Then we will prove that the set $P_n = A_1 \cup A_2 \cup A_3$ is also complete and will satisfy the conditions i) and ii), where $A_1 = P_{n_1}P_{n_2}B_{n_3}$, $A_2 = P_{n_1}B_{n_2}P_{n_3}$, $A_3 = B_{n_1}P_{n_2}P_{n_3}$ and $n = \sum_{j=1}^{3} n_j$.

Clearly, the sets A_1, A_2 and A_3 are pairwise disjoint and each of them satisfies the condition i). The constructions of the sets A_1, A_2 and A_3 imply that every pair A_i, A_j has one common P_{n_k} in the same position for some $n_k \in \{n_1, n_2, n_3\}$, therefore the set $P_n = A_1 \cup A_2 \cup A_3$ satisfies the condition i).

We have to prove by contradiction that the set A_1 will satisfy the condition ii). Assume that there are three pairwise distinct points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n) \in A_1$ so that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. Then $\boldsymbol{\alpha}^{(i)} + \boldsymbol{\beta}^{(i)} + \boldsymbol{\gamma}^{(i)} \equiv 0(mod\ 3)$, where $\boldsymbol{\alpha}^{(i)} = \left(\alpha_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \alpha_{\sum_{j=1}^{i} n_j}\right)$, $\boldsymbol{\beta}^{(i)} = \left(\beta_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \beta_{\sum_{j=1}^{i} n_j}\right)$, $\boldsymbol{\gamma}^{(i)} = \left(\gamma_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \gamma_{\sum_{j=1}^{i} n_j}\right)$ and $i = 1, 2, 3$. Since, by induction hypothesis, both P_{n_1} and P_{n_2} satisfy the condition ii), therefore $\boldsymbol{\alpha}^{(i)} = \boldsymbol{\beta}^{(i)} = \boldsymbol{\gamma}^{(i)}$, where $i = 1, 2$. Claim 1 implies that $\boldsymbol{\alpha}^{(3)} = \boldsymbol{\beta}^{(3)} = \boldsymbol{\gamma}^{(3)}$. Hence $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$, which contradicts our assumption. By a similar arguments, one can prove that the sets A_2 and A_3 also satisfy the condition ii).

Now we will prove that the set $P_n = A_1 \cup A_2 \cup A_3$ also satisfies the condition ii). Assume that there are three pairwise distinct points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n) \in P_n$ so that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. Since we have already proved that the points $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ cannot belong to the same set A_i, $1 \le i \le 3$, thereby only the following two cases are possible.

Case 1. Each point belongs to only one set, say $\boldsymbol{\alpha} \in A_1$, $\boldsymbol{\beta} \in A_2$ and $\boldsymbol{\gamma} \in A_3$. By construction of the sets A_1 and A_2 both $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_{n_1})$ and $\boldsymbol{\beta}^{(1)} = (\beta_1, \cdots, \beta_{n_1})$ belong to P_{n_1}. Hence, by definition of P_{n_1} there exists i, $1 \le i \le n_1$, so that $\alpha_i = \beta_i = 0$. Since $\boldsymbol{\gamma}^{(1)} = (\gamma_1, \cdots, \gamma_{n_1}) \in B_{n_1}$, therefore $\gamma_i = 1\ or\ 2$. Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0(mod\ 3)$, which contradicts the assumption that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$.

Case 2. Only two points from $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ belong to the same set, say $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_1$ and $\boldsymbol{\gamma} \in A_2$. Then $\boldsymbol{\alpha}^{(2)} + \boldsymbol{\beta}^{(2)} + \boldsymbol{\gamma}^{(2)} \not\equiv 0(mod 3)$. Since, by construction of the set A_1, $\boldsymbol{\alpha}^{(2)} = (\alpha_{n_1+1}, \cdots, \alpha_{n_1+n_2})$ and $\boldsymbol{\beta}^{(2)} = (\beta_{n_1+1}, \cdots, \beta_{n_1+n_2}) \in P_{n_2}$, hence there is i so that $\alpha_i = \beta_i = 0, n_1 + 1 \le i \le n_1 + n_2$. But by construction of the set A_2, $\gamma^{(2)} \in B_{n_2}$, therefore $\gamma_i = 1\ or\ 2$. Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0(mod\ 3)$, which, again, contradicts the assumption that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. So, P_n satisfies the condition ii).

We will prove the completeness of P_n again by contradiction. Let us assume that there is a point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, so that $\boldsymbol{\alpha} \notin P_n$ and $P_n \cup \{\boldsymbol{\alpha}\}$ satisfies the conditions i) and ii). Let's represent the point $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{\alpha}^{(3)}$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_{n_1})$, $\boldsymbol{\alpha}^{(2)} = (\alpha_{n_1+1}, \cdots, \alpha_{n_1+n_2})$ and $\boldsymbol{\alpha}^{(3)} = (\alpha_{n_1+n_2+1}, \cdots, \alpha_{n_1+n_2+n_3})$. Condition i) for the set $P_n \cup \{\boldsymbol{\alpha}\}$ follows that at least two of the following three sets $P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$, $P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$, $P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}$ must satisfy the condition i). Since, otherwise, if at most only one of them, say $P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$, satisfies the condition i), one can find points $\boldsymbol{\delta} = (\delta_{n_1+1}, \cdots, \delta_{n_1+n_2}) \in P_{n_2}$ and $\boldsymbol{\theta} = (\theta_{n_1+n_2+1}, \cdots, \theta_{n_1+n_2+n_3}) \in P_{n_3}$, so that the points $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{\alpha}^{(3)}$ and $\boldsymbol{x} = \boldsymbol{\vartheta\delta\theta} \in P_n$ will not satisfy the condition i), for every point $\boldsymbol{\vartheta}$ from B_{n_1}. By a similar argument one can prove that $P_n \cup \{\boldsymbol{\alpha}\}$ does not satisfy the condition i), when only $P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$ or only $P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}$ satisfy the condition i). Therefore, at least two of the following three sets $P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$, $P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$ and $P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}$ satisfy the condition i). If the sets $P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$ and $P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$ satisfy the condition i), then, by induction hypothesis, the completeness of P_{n_1} and P_{n_2} implies that $\boldsymbol{\alpha}^{(1)}$ coincides with one of the points of P_{n_1} and $\boldsymbol{\alpha}^{(2)}$ coincides with one of the points of P_{n_2}, respectively. If $\boldsymbol{\alpha}^{(3)} \notin B_{n_3}$ we will choose $\boldsymbol{x}^{(3)} = (x_{n_1+n_2+1}, \cdots, x_{n_1+n_2+n_3})$ and $\boldsymbol{y}^{(3)} = (y_{n_1+n_2+1}, \cdots, y_{n_1+n_2+n_3})$ from B_{n_3} in the following way: if $\alpha_i = 0$ then $x_i = 1$ and $y_i = 2$, otherwise, $x_i = y_i = \alpha_i$, $n_1 + n_2 + 1 \le i \le n_{n_1+n_2+n_3}$. Then $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{x}^{(3)}$, $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{y}^{(3)} \in P_n$ and Claim 1 implies that $\boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{\alpha}^{(3)} + \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{x}^{(3)} + \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{y}^{(3)} \equiv 0(mod\ 3)$, which contradicts the fact that $P_n \cup \{\boldsymbol{\alpha}\}$ satisfies the condition ii). Hence $\boldsymbol{\alpha}^{(3)} \in B_{n_3}$ and therefore $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)}\boldsymbol{\alpha}^{(3)} \in P_n$, which contradicts the assumption that $\boldsymbol{\alpha} \notin P_n$.

Having the sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}, P_{n_6}$ and $B_{n_1}, B_{n_2}, B_{n_3}, B_{n_4}, B_{n_5}, B_{n_6}$, by concatenation of the points of the sets one can form $C_6^3 = 20$ combinations with three P_{n_i}'s and three B_{n_i}'s, ten of them which that we need are listed below: $A_1 = P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6}$, $A_2 = B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6}$, $A_3 = P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6}$, $A_4 = B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6}$, $A_5 = B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_6 = B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$, $A_7 = B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6}$, $A_8 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}$, $A_9 = P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6}$, $A_{10} = P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6}$.

Notice that each one of the other ten combinations is a complement of some other listed above in the sense that can be formed by replacing P_{n_i}'s by B_{n_i}'s and conversely, by replacing B_{n_i}'s by P_{n_i}'s.

Theorem 2[11]. The following recurrence relation $P_n = \bigcup_{i=1}^{10} A_i$, with initial sets $P_1 = \{(0)\}$, $P_2 = \{(0, 1), (0, 2)\}$ and $n = \sum_{i=1}^{6} n_i$ yields a complete P_n set.

Proof. We prove the theorem using induction on n. As mentioned in Theorem 1. $P_1 = \{(0)\}$, $P_2 = \{(0,1),(0,2)\}$ and

$P_3 = \{(0,0,1),(0,0,2),(0,1,0),(0,2,0),(1,0,0),(2,0,0)\}$ are complete and they satisfies conditions i) and ii). Assume that the sets $P_{n_1}, P_{n_2}, P_{n_3}, P_{n_4}, P_{n_5}$ and P_{n_6} are complete and satisfy the conditions i) and ii). Then we will prove that the set $P_n = \bigcup_{i=1}^{10} A_i$ is also complete and satisfies the conditions i) and ii), where $n = \sum_{i=1}^{6} n_i$. Clearly, the sets $A_1, A_2, \ldots, A_{10}$ are pairwise disjoint and each of them satisfies the condition i). The constructions of the sets $A_1, A_2, \ldots, A_{10}$ imply that every pair A_i, A_j ($i \neq j$, i, j $\in \{1, 2, \ldots, 10\}$) has at least one common P_{n_k} in the same position for some $n_k \in \{n_1, n_2, n_3, n_4, n_5, n_6\}$, therefore the set $P_n = \bigcup_{i=1}^{10} A_i$ satisfies the condition i).

We have to prove by contradiction that the set A_1 will satisfy the condition ii). Assume that there are three pairwise distinct points $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n) \in A_1$ so that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. Then $\boldsymbol{\alpha}^{(i)} + \boldsymbol{\beta}^{(i)} + \boldsymbol{\gamma}^{(i)} \equiv 0(mod\ 3)$, where $\boldsymbol{\alpha}^{(i)} = \left(\alpha_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \alpha_{\sum_{j=1}^{i} n_j}\right)$, $\boldsymbol{\beta}^{(i)} = \left(\beta_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \beta_{\sum_{j=1}^{i} n_j}\right)$, $\boldsymbol{\gamma}^{(i)} = \left(\gamma_{\sum_{j=1}^{i-1} n_j+1}, \cdots, \gamma_{\sum_{j=1}^{i} n_j}\right)$ and $i \in \{1, 2, \dots, 6\}$. Since, by induction hypothesis P_{n_1}, P_{n_2} and P_{n_6}satisfy the condition ii), therefore $\boldsymbol{\alpha}^{(1)} = \boldsymbol{\beta}^{(1)} = \boldsymbol{\gamma}^{(1)}$, $\boldsymbol{\alpha}^{(2)} = \boldsymbol{\beta}^{(2)} = \boldsymbol{\gamma}^{(2)}$ and $\boldsymbol{\alpha}^{(6)} = \boldsymbol{\beta}^{(6)} = \boldsymbol{\gamma}^{(6)}$**.** Claim 1 implies that $\boldsymbol{\alpha}^{(i)} = \boldsymbol{\beta}^{(i)} = \boldsymbol{\gamma}^{(i)}$, $i = 3, 4, 5$**.** Therefore $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$, which contradicts our assumption. In the same way, one can prove that the set A_i, $i \in \{2, \dots, 10\}$**,** also satisfies the condition ii).

Now we have to prove that the set $P_n = \bigcup_{i=1}^{10} A_i$ also satisfies the condition ii). Assume that there are three pairwise distinct points $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)$ and $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n) \in P_n$ so that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. Since we have already proved that the points $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ cannot belong to the same set A_i, $i \in \{1, 2, \dots, 10\}$, thereby the following two cases are possible.

Case 1. Each point belongs to only one set, say $\boldsymbol{\alpha} \in A_1$, $\boldsymbol{\beta} \in A_2$ and $\boldsymbol{\gamma} \in A_3$. By constructions of the sets A_1 and A_3 both $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_{n_1})$ and $\boldsymbol{\gamma}^{(1)} = (\gamma_1, \cdots, \gamma_{n_1})$ belong to P_{n_1}. Hence, by definition of P_{n_1} there exists i, $1 \le i \le n_1$, so that $\alpha_i = \gamma_i = 0$. Since $\boldsymbol{\beta}^{(1)} = (\beta_1, \cdots, \beta_{n_1}) \in B_{n_1}$, we have $\beta_i = 1\ or\ 2$. Hence $\alpha_i + \beta_i + \gamma_i \not\equiv 0(mod\ 3)$, which contradicts our assumption that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$.

Case 2. Only two points from $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ belong to the same set, say $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in A_1$ and $\boldsymbol{\gamma} \in A_2$. Since $\boldsymbol{\alpha}^{(1)}$, $\boldsymbol{\beta}^{(1)} \in P_{n_1}$, by definition of P_{n_1}there is i, so that $\alpha_i = \beta_i = 0, 1 \le i \le n_1$. But $\boldsymbol{\gamma}^{(1)} = (\gamma_{n_1+1}, \cdots, \gamma_{n_1+n_2}) \in B_{n_1}$, hence $\gamma_i = 1,\ 2$. Therefore $\alpha_i + \beta_i + \gamma_i \not\equiv 0(mod\ 3)$, which, again, contradicts the assumption that $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv 0(mod\ 3)$. The proof for all other possible cases is similar to the one given above. So, P_n satisfies the condition ii).

We will prove the completeness of P_n again by contradiction. Let us assume that there is a point $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$, so that $\boldsymbol{\alpha} \notin P_n$ and $P_n \cup \{\boldsymbol{\alpha}\}$ satisfies the conditions i) and ii). Let's represent $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)} \cdots \boldsymbol{\alpha}^{(6)}$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, \cdots, \alpha_{n_1})$, $\boldsymbol{\alpha}^{(2)} = (\alpha_{n_1+1}, \cdots, \alpha_{n_1+n_2})$, $\cdots$, $\boldsymbol{\alpha}^{(6)} = \left(\alpha_{\sum_{j=1}^{5} n_j+1}, \cdots, \alpha_{\sum_{j=1}^{6} n_j}\right)$. Now we have to prove that condition i) for the set $P_n \cup \{\boldsymbol{\alpha}\}$ implies that there is $i_0 \in \{1, 2, \cdots, 10\}$ so that the three sets $C_{i_0,1}$, $C_{i_0,2}$ and $C_{i_0,3}$ of the following ten families of sets

$C_{1,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C_{1,2} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C_{1,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C_{2,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C_{2,2} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C_{2,3} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}$,
$C_{3,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C_{3,2} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C_{3,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C_{4,1} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C_{4,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C_{4,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C_{5,1} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C_{5,2} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}, C_{5,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C_{6,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C_{6,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C_{6,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C_{7,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C_{7,2} = P_{n_5,} \cup \{\boldsymbol{\alpha}^{(5)}\}, C_{7,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C_{8,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C_{8,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C_{8,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C_{9,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C_{9,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C_{9,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C_{10,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C_{10,2} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C_{10,3} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}$.

must satisfy the condition i), where $i_0 \in \{1, 2, \dots, 10\}$. Otherwise, the following two cases are possible:

Case 1. There is $i_0 \in \{1, 2, \dots, 10\}$, so that only the following three sets $C'_{i_0,1}$, $C'_{i_0,2}$ and $C'_{i_0,3}$ satisfy the condition i). Note that all such ten family of sets are listed below:

$C'_{1,1} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C'_{1,2} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}, C'_{1,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C'_{2,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C'_{2,2} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}, C'_{2,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C'_{3,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C'_{3,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C'_{3,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C'_{4,1} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C'_{4,2} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}, C'_{4,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C'_{5,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C'_{5,2} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C'_{5,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C'_{6,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C'_{6,2} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C'_{6,3} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}$,
$C'_{7,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C'_{7,2} = P_{n_3,} \cup \{\boldsymbol{\alpha}^{(3)}\}, C'_{7,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$,
$C'_{8,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}, C'_{8,2} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C'_{8,3} = P_{n_4} \cup \{\boldsymbol{\alpha}^{(4)}\}$,
$C'_{9,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C'_{9,2} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C'_{9,3} = P_{n_5} \cup \{\boldsymbol{\alpha}^{(5)}\}$,
$C'_{10,1} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}, C'_{10,2} = P_{n_3} \cup \{\boldsymbol{\alpha}^{(3)}\}, C'_{10,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$.

Assume that $i_0 = 1$. One can find points $\boldsymbol{\delta_1} = (\delta_1, \cdots, \delta_{n_1}) \in P_{n_1}$, $\boldsymbol{\delta_2} = (\delta_{n_1+1}, \cdots, \delta_{n_1+n_2}) \in P_{n_2}$, and $\boldsymbol{\delta_3} = (\delta_{n_1+n_2+1}, \cdots, \delta_{n_1+n_2+n_3}) \in P_{n_3}$, so that points $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)} \cdots \boldsymbol{\alpha}^{(6)}$ and $\boldsymbol{\delta_1\delta_2\delta_3x_4x_5x_6} \in P_n$, do not satisfy the condition i), for every $\boldsymbol{x_4} \in B_{n_4}$, $\boldsymbol{x_5} \in B_{n_5}$ and $\boldsymbol{x_6} \in B_{n_6}$.

Case 2. At most two sets $P_{n_i} \cup \{\boldsymbol{\alpha}^{(i)}\}$ and $P_{n_j} \cup \{\boldsymbol{\alpha}^{(j)}\}$ satisfy the condition i), where $i \neq j, i$, j $\in \{1, 2, \dots, 6\}$.

Assume, for instance, that $P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$ and $P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$ satisfy condition i). Then, there are points $\boldsymbol{\delta_3} = \left(\delta_{\sum_{j=1}^{2} n_j+1}, \cdots, \delta_{\sum_{j=1}^{3} n_j}\right) \in P_{n_3}$, $\boldsymbol{\delta_5} = \left(\delta_{\sum_{j=1}^{4} n_j+1}, \cdots, \delta_{\sum_{j=1}^{5} n_j}\right) \in P_{n_5}$, and $\boldsymbol{\delta_6} = \left(\delta_{\sum_{j=1}^{5} n_j+1}, \cdots, \delta_{\sum_{j=1}^{6} n_j}\right) \in P_{n_6}$, so that points $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(1)}\boldsymbol{\alpha}^{(2)} \cdots \boldsymbol{\alpha}^{(6)}and\boldsymbol{x} = \boldsymbol{x_1x_2\delta_3x_4\delta_5\delta_6} \in P_n$, will not satisfy the condition i), for every $\boldsymbol{x_1} \in B_{n_1}$, $\boldsymbol{x_2} \in B_{n_2}$, $\boldsymbol{x_4} \in B_{n_4}$. The other cases are similar.

Suppose that the following three sets $C_{1,1} = P_{n_1} \cup \{\boldsymbol{\alpha}^{(1)}\}$**,** $C_{1,2} = P_{n_2} \cup \{\boldsymbol{\alpha}^{(2)}\}$ and $C_{1,3} = P_{n_6} \cup \{\boldsymbol{\alpha}^{(6)}\}$ satisfy the condition i). Then, by induction hypothesis the completeness of P_{n_1}**,** P_{n_2} and P_{n_6} implies that $\boldsymbol{\alpha}^{(1)}$ coincides with one of the points of P_{n_1}, $\boldsymbol{\alpha}^{(2)}$ coincides with one of the points of P_{n_2} and $\boldsymbol{\alpha}^{(6)}$ coincides with one of the points of P_{n_6}, respectively. When $\boldsymbol{\alpha}^{(i)} \notin B_{n_i}$, we will choose two distinct points $\boldsymbol{\mu_i^j} = \left(\mu^j_{\sum_{k=1}^{i-1} n_k+1}, \cdots, \mu^j_{\sum_{k=1}^{i} n_k}\right)$ from B_{n_i} in the following way: if $\alpha_l = 0$ then $\mu_l^1 = 1$ and $\mu_l^2 = 2$, otherwise, $\mu_l^1 = \mu_l^2 = \alpha_l$, $\sum_{k=1}^{i-1} n_k + 1 \le l \le \sum_{k=1}^{i} n_k$, where $i \in \{3, 4, 5\}$ and $j = 1, 2$. In the case when $\boldsymbol{\alpha}^{(i)} \in B_{n_i}$

we will assume that $\mu_i^1 = \mu_i^2 = \alpha^{(i)}$. Then Claim 1 implies that $\alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(6)} + \alpha^{(1)}\alpha^{(2)}\mu_3^1\mu_4^1\mu_5^1\alpha^{(6)} + \alpha^{(1)}\alpha^{(2)}\mu_3^2\mu_4^2\mu_5^2\alpha^{(6)} \equiv 0(mod\ 3)$, which contradicts that $P_n \cup \{\alpha\}$ satisfies the condition ii). Hence $\alpha^{(i)} \in B_{n_i}$ ($i = 3, 4, 5$) and therefore $\alpha = \alpha^{(1)}\alpha^{(2)}\cdots\alpha^{(6)} \in P_n$, which contradicts the assumption that $\alpha \notin P_n$. The proof for all other three sets $C_{i,1}$, $C_{i,1}$ and $C_{i,3}$ is similar to one described above, where $i \in \{2, \dots, 10\}$.

Claim 2. Note that from the construction of P_n in both theorems it follows that if the point $p = (p_1, \dots, p_i, \dots, p_n) \in P_n$ and $p_i \neq 0$, then, also, the point $p' = (p_1, \dots, p_i^{-1}, \dots, p_n) \in P_n$, where p_i^{-1} is the opposite number of p_i in the field F_3.

Theorem 3. If P_n is constructed by Theorem 1 or Theorem 2, then $S_n' = P_n\{1\} \cup B_n'\{0\}$ is a complete cap in n-dimensional projective geometry PG$(n, 3)$.

Proof. First we will prove that the set $S_n' = P_n\{1\} \cup B_n'\{0\}$ is a cap. Suppose that S_n' is not a cap. Then there are a triple of distinct points $x, y, z \in S_n'$ and numbers k, l, $m \in \{1, 2\}$, so that $kx + ly + mz \equiv 0(mod\ 3)$, where $x = (x_1, \dots, x_n, x_{n+1})$, $y = (y_1, \dots, y_n, y_{n+1})$ and $z = (z_1, \dots, z_n, z_{n+1})$. The following four cases are possible.

Case 1. x, y, $z \in P_n\{1\}$. Therefore $kx_{n+1} + ly_{n+1} + mz_{n+1} \equiv 0(mod\ 3)$. Hence $k + l + m = 0$. Claim 1 implies that $k = l = m$. Hence $x' + y' + z' \equiv 0(mod\ 3)$, where $x' = (x_1, \dots, x_n)$, $y' = (y_1, \dots, y_n)$ and $z' = (z_1, \dots, z_n)$. Therefore points x', y' and z' are collinear in AG(n, 3), which is impossible by definition of P_n.

Case 2. x, y and $z \in B_n'\{0\}$. Claim 1 implies that $x' = y' = z'$, where $x' = (x_1, \dots, x_n)$, $y' = (y_1, \dots, y_n)$ and $z' = (z_1, \dots, z_n)$, which again is impossible.

Case 3. Two of the points, say $x, y \in P_n\{1\}$ and $z \in B_n'\{0\}$. Then there is i, $1 \leq i \leq n$, so that $x_i = y_i = 0$, but $z_i \neq 0$. Then the points x, y and z are not collinear.

Case 4. Two of the points, say x, $y \in B_n'\{0\}$ and $z \in P_n\{1\}$. Then x, y and z are not collinear by their last coordinate.

We will prove the completeness of S_n', also, by contradiction. Assume that there is a point $\alpha = (\alpha_1, \dots, \alpha_{n+1})$, so that $\alpha \notin S_n'$ and $S_n' \cup \{\alpha\}$ is a cap in PG(n, 3). If $\alpha_{n+1} = 0$, then one can choose two points $x = (x_1, \dots, x_n, 0)$ and $y = (y_1, \dots, y_n, 0) \in B_n'\{0\}$ so that: if $\alpha_i \neq 0$, then $x_i = y_i = \alpha_i$, else $x_i = 1$, $y_i = 2$, $1 \leq i \leq n$. Obviously, points $x, y,$ and α are collinear. Otherwise one can assume that $\alpha_{n+1} = 2$. Then the completeness of P_n follows that $P_n \cup \alpha'$ does not satisfy the condition i), where $\alpha' = (\alpha_1, \dots, \alpha_n)$. Therefore there is a point $\beta = (\beta_1, \dots, \beta_n) \in P_n$ so that, if $\beta_i = 0$, then $\alpha_i \neq 0$ and conversely if $\alpha_i = 0$, then $\beta_i \neq 0$, $1 \leq i \leq n$. Using Claim 1 and Claim 2, one can choose the point $\gamma = (\gamma_1, \dots, \gamma_n, 0) \in B_n'\{0\}$ so that $\alpha' + \beta + \gamma' \equiv 0(\text{mod } 3)$, where $\gamma' = (\gamma_1, \dots, \gamma_n)$.

Corollary. For every natural number n, $s_{n,3}' \geq |P_n| + 2^{n-1}$.

Notice that the cardinality of P_n obtained by Theorem 1 (Theorem 2), essentially depends on the representation of n as the sum of three (six) natural numbers. Presenting the natural numbers as the sum of six natural numbers and applying Theorem 2, for some $n \geq 6$ in some cases, one can obtain larger complete P_n sets than those, which are constructed by Theorem 1. It is obvious that $|P_1| = 1$, $|P_2| = 2$, and $|P_{1+1+1}| = 6$. $|P_{2+1+1}| = 12$, $|P_{3+1+1}| = 32$, $|P_{1+1+1+1+1+1}| = 80$, $|P_9| = |P_{3+3+3}| = 864$. Therefore, the corollary implies that $s_{3,3}' \geq 10$, $s_{4,3}' \geq 20$, $s_{5,3}' \geq 48$, $s_{6,3}' \geq 112$. So the lower bounds obtained by Theorem 3 for n = $3, 4, 6$ are sharp and $s_{9,3}' \geq 1120$. Since $|P_{2+2+1}| = 24$ and $|P_{2+2+2}| = 48$, the corollary implies that the sizes of the least complete caps in PG(5, 3) and PG(6, 3) are not greater than 40 and 80, respectively.

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