Ore-type Conditions for Hamilton Cycles and Spanning Trees with Few Leaves

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***Abstract*— A leaf of a tree is a vertex with degree one. For a given integer k, a tree with at most k leaves is called a k-ended tree. We present two new Ore-type conditions for Hamilton cycles and k-ended spanning trees.**

***Keywords—Hamilton cycle*, leaf, k-ended spanning tree, Ore-type condition.**

I. Introduction

We consider only finite, undirected graphs with no loops or multiple edges. A good reference for any undefined terms is [1]. Let $\mathrm{G} = (\mathrm{V(G)}, \mathrm{E(G)})$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex subset $X \subseteq V(G)$, we use $|X|$ to denote the cardinality of X. For a vertex $v \in V(G)$, we denote by $N(v)$ the neighborhood of v in G, $d(v) = |N(v)|$ the degree of v in G.

A path (simple path) of order m, denoted by P_m, is a sequence of different vertices $v_1, \dots, v_m$, denoted by $v_1 v_2 \dots v_m$, such that $v_{i-1} v_i \in E(G)$ for all $2 \le i \le m$. Similarly, a cycle (simple cycle) of order m is a sequence of distinct vertices $v_1, \dots, v_m$, denoted by $v_1 v_2 \dots v_m v_1$, such that $v_{i-1} v_i \in E(G)$ and $v_m v_1 \in E(G)$ for all $2 \le i \le m$. In particular, if $m = 2$, then $v_1 v_2 v_1$ is a cycle of order 2; and if $m = 1$, then $v_1 v_1$ is a cycle of order 1. So, by the definition, every vertex (edge) can be interpreted as a cycle of order 1 (2, respectively). A graph G is Hamiltonian, if it contains a Hamilton cycle - a simple spanning cycle. This extension of the cycle concept allows us to avoid undesirable additional condition $n \ge 3$ in almost all Hamiltonian sufficient conditions for graphs on n vertices.

A leaf of a tree is a vertex with degree one. A tree with at most k leaves is called a k-ended tree, where $k \ge 2$ is an integer.

Suppose that all the edges of a path $P = v_1 v_2 \dots v_m$ are directed from v_1 to v_m. For $x \in \{v_2, \dots, v_m\}$, we denote the predecessor of x on P by x^-. For a subset $U \subseteq V(P) - \{v_1\}$, we denote $U^- = \{u^- : u \in U\}$.

The first sufficient condition for a graph to be Hamiltonian was given by Dirac [3] in terms of minimum degree δ.

Theorem A (Dirac [3], 1952). Every graph on n vertices with $\delta \ge \frac{n}{2}$ is Hamiltonian.

In [5], Ore relaxed the minimum degree condition in Theorem A in terms of degree sums of nonadjacent vertices.

Theorem B (Ore [5], 1961). Every graph on n vertices is Hamiltonian if $d(u) + d(v) \ge n$ for each nonadjacent vertices u and v.

In 1976, an analogous condition was given for k-ended spanning trees [4] in connected graphs.

Theorem C (Nikoghosyan [4], 1976). Let G be a connected graph on n vertices and let $k \ge 2$ be an integer. If $d(u) + d(v) \ge n - k + 1$ for each nonadjacent vertices u, v, then G has a k-ended spanning tree.

Actually, Theorem C is a starting point for all analogous results concerning spanning trees with few leaves and branch vertices (a branch vertex of a tree is a vertex with degree more than two). PDF-file of article [4] is available at http://ysu.am/files/1-1599118662-.pdf. Theorem C is also mentioned in [6]. However, the literature mainly refers to the work of Broersma and Tuinstra [2], published in 1998 and having 39 citations in Scopus.

In this paper, we present two new Ore-type conditions for Hamilton cycles and k-ended spanning trees by relaxing the conditions in Theorems B and C on special induced subgraphs in terms of K_m (complete graph), $K_{m,n}$ (complete bipartite graph), P_m (simple path on m vertices) and $W + e$ (adding an edge e to a subgraph W).

Theorem 1. Let G be a graph on $n \ge 4$ vertices satisfying $d(x) + d(y) \ge n$ for each nonadjacent vertices x, y of every induced $K_{1,2} \cup K_1$, $K_3 \cup K_1$, $K_{1,3}$, $K_{1,3} + e$, P_4. If G has a vertex v with $d(v) \ge 2$, then G is Hamiltonian.

Theorem 2. Let G be a connected graph on n vertices and let $k \ge 2$ be an integer. If $d(x) + d(y) \ge n - k + 1$ for each nonadjacent vertices x, y of every induced $K_{1,2} \cup K_1$, $K_3 \cup K_1$, $K_{1,3}$, $K_{1,3} + e$, P_4, then G has a spanning k-ended tree.

Observing that the graphs $K_3 \cup K_1$, $K_{1,3}$ and P_4 are the special cases of $(K_{1,2} \cup K_1) + e$, we can formulate the following two corollaries.

Corollary 1. Let G be a graph on $n \geq 4$ vertices satisfying $d(x) + d(y) \geq n$ for each nonadjacent vertices x, y of every induced $K_{1,2} \cup K_1$, $(K_{1,2} \cup K_1) + e$ and $K_{1,3} + e$. If G has a vertex v with $d(v) \geq 2$, then G is Hamiltonian.

Corollary 2. Let G be a connected graph on n vertices and let $k \geq 2$ be an integer. If $d(x) + d(y) \geq n - k + 1$ for each nonadjacent vertices x, y of every induced $K_{1,2} \cup K_1$, $(K_{1,2} \cup K_1) + e$ and $K_{1,3} + e$, then G has a spanning k-ended tree.

II. PROOFS

Proof of Theorem 1. Assume first that G is not connected and let $H_1, H_2, \ldots, H_h$ be the connected components of G. By the hypothesis, G has a vertex v such that $d(v) \geq 2$. Assume without loss of generality that $v \in V(H_1)$. Then we can form a path $Q = xvy$ in H_1. Let $z \in V(H_2)$ and let F be a graph induced on $\{x, y, z, v\}$. If $xy \notin E(G)$, then clearly $F = K_{1,2} \cup K_1$. Otherwise, $F = K_3 \cup K_1$. In both cases we have $vz \notin E(F)$. By the hypothesis, $d(v) + d(z) \geq n$, implying that

$$n \geq |V(H_1)| + |V(H_2)|$$
$$\geq (d(v) + 1) + (d(z) + 1) \geq n + 2,$$

a contradiction. So, G is connected.

Let $P = x_1 x_2 \ldots x_p$ be a longest path in G.

Case 1. $x_1 x_p \in E(G)$.

Put $C = x_1 x_2 \ldots x_p x_1$. If $p = n$, then C is a Hamilton cycle in G and we are done. Otherwise, since G is connected, there is an edge $y_1 y_2 \in E(G)$ such that $y_1 \in V(C)$ and $y_2 \notin V(C)$. Choose a vertex $y_3 \in V(C)$ such that $y_1 y_3 \in E(C)$. Then $C - y_1 y_3 + y_1 y_2$ is a path longer than P, a contradiction.

Case 2. $x_1 x_p \notin E(G)$.

Case 2.1. Either $d(x_1) \geq 2$ or $d(x_p) \geq 2$.

Assume without loss of generality that $d(x_1) \geq 2$. Recalling that P is a longest path in G, we have $N(x_1) \subset V(P)$. Then we can choose $i \geq 3$ such that $x_1 x_i \in E(G)$.

Case 2.1.1. $x_{i-1} x_p \in E(G)$.

Put $C_1 = x_1 x_2 \ldots x_{i-1} x_p x_{p-1} \ldots x_i x_1$. If $p = n$, then C_1 is a Hamilton cycle in G. If $p \neq n$, then we can argue as in Case 1.

Case 2.1.2. $x_{i-1} x_p \notin E(G)$.

Since $i \geq 3$, we have $x_{i-1} \neq x_1$, implying that x_1, x_{i-1}, x_i, x_p are distinct vertices. Let R be a graph induced on $\{x_1, x_{i-1}, x_i, x_p\}$. If $x_1 x_{i-1} \notin E(G)$ and $x_i x_p \notin E(G)$, then $R = K_{1,2} \cup K_1$. Next, if $x_1 x_{i-1} \notin E(G)$ and $x_i x_p \in E(G)$, then $R = K_{1,3}$. Further, if $x_1 x_{i-1} \in E(G)$ and $x_i x_p \notin E(G)$, then $R = K_3 \cup K_1$. Finally, if $x_1 x_{i-1} \in E(G)$ and $x_i x_p \in E(G)$, then $R = K_{1,3} + x_i x_p$. Since $x_1 x_p \notin E(R)$, by the hypothesis, $d(x_1) + d(x_p) \geq n$. If $N^-(x_1) \cap N(x_p) \neq \emptyset$, then we can argue as in Case 2.1.1. Let $N^-(x_1) \cap N(x_p) = \emptyset$. Observing that $x_p \notin N^-(x_1) \cup N(x_p)$, we obtain

$$p \geq |N^-(x_1)| + |N(x_p)| + |\{x_p\}|$$
$$\geq d(x_1) + d(x_p) + 1 \geq n + 1,$$

a contradiction.

Case 2.2. $d(x_1) = d(x_p) = 1$.

Let R be a graph induced by $\{x_1, x_2, x_3, x_p\}$. If $p \geq 5$, then $R = K_{1,2} \cup K_1$. If $p = 4$, then $R = P_4$. Now let $p = 3$. Since $n \geq 4$ and G is connected, we have $yx_2 \in E(G)$ for some vertex $y \notin V(P)$. Since P is a longest path in G, $\{y, x_1, x_3\}$ is an independent set of vertices. Hence, $R = K_{1,3}$. By the hypothesis, $2 = d(x_1) + d(x_3) \geq n$, a contradiction. Theorem 1 is proved. ■

Proof of Theorem 2. Let $P = x_1 x_2 \ldots x_p$ be a longest path in G.

Case 1. $p \geq n - k + 2$.

Since G is connected, we can form a spanning tree T in G including P as a subpath with end vertices x_1 and x_p. Clearly, T has at most

$$|V(G)| - |V(P) \backslash \{x_1, x_p\}| = n - p + 2 \leq k$$

leaves and we are done.

Case 2. $p \leq n - k + 1$.

If $p = n$, then P is a Hamilton path in G. This means that G has an r-ended spanning tree with $r = 2 \leq k$ and we are done. Now let $p \neq n$. If $x_1 x_p \in E(G)$, then recalling that G is connected, we can form a path longer than P as in proof of Theorem 1 (Case 1), a contradiction. So, we can assume that $x_1 x_p \notin E(G)$.

Case 2.1. Either $d(x_1) \geq 2$ or $d(x_p) \geq 2$.

Assume without loss of generality that $d(x_1) \geq 2$. Recalling that P is a longest path in G, we have $N(x_1) \subset V(P)$. Choose $i \geq 3$ such that $x_1 x_i \in E(G)$. If $x_{i-1} x_p \in E(G)$, then using the cycle $x_1 x_2 \ldots x_{i-1} x_p x_{p-1} \ldots x_i x_1$ and recalling that $p \neq n$, we can form a path longer than P as in proof of Theorem 1 (Case 1). Let $x_{i-1} x_p \notin E(G)$. Clearly, x_1, x_{i-1}, x_i, x_p are distinct vertices. Let R be a graph induced on $\{x_1, x_{i-1}, x_i, x_p\}$. As in proof of Theorem 1 (Case 2.1.2), we have either $R = K_{1,2} \cup K_1$ or $R = K_{1,3}$ or $R = K_3 \cup K_1$ or $R = K_{1,3} + x_i x_p$. By the hypothesis, $d(x_1) + d(x_p) \geq n$. If $N^-(x_1) \cap N(x_p) \neq \emptyset$, then we can argue as in Case 2.1.1. Let $N^-(x_1) \cap N(x_p) = \emptyset$. Observing that $x_p \notin N^-(x_1) \cup N(x_p)$, we obtain

$$p \geq |N^-(x_1)| + |N(x_p)| + |\{x_p\}|$$
$$\geq d(x_1) + d(x_2) + 1 \geq n + 1,$$

a contradiction.

Case 2.2. $d(x_1) = d(x_p) = 1$.

Case 2.2.1. $2 \leq p \leq 3$.

If $p = 2$, then clearly $n = p = 2$. This means that P is a 2-ended spanning tree in G and we are done. Hence $p = 3$. If $n = 3$, then again P is a 2-ended spanning tree in G. Now let n≥ 4. Since G is connected and $d(x_1) = d(x_p) = 1$, we have $yx_2 \in E(G)$ for some vertex $y \notin V(P)$. Since P is a longest path in G, $\{y, x_1, x_3\}$ is an independent set of vertices and $R = K_{1,3}$. By the hypothesis, $2 = d(x_1) + d(x_3) \geq n \geq 4$, a contradiction.

Case 2.2.2. $p \geq 4$.

Let R be a graph induced by $\{x_1, x_2, x_3, x_p\}$. If $p = 4$, then $R = P = P_4$. If $p \geq 5$, then $R = K_{1,2} \cup K_1$. By the hypothesis, $2 = d(x_1) + d(x_p) \geq n \geq 4$, a contradiction. Theorem 2 is proved. ■

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