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Asymptotics for the Logarithm of the Number of k-Solution-Free Collections in Abelian Groups

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***Abstract*—A collection $(A_1, \dots, A_k)$ of subsets of a group G is called k-solution-free if the equation $x_1 + \dots + x_k = 0$ has no solution in $(A_1, \dots, A_k)$, where $x_1 \in A_1, \dots, x_k \in A_k$. The asymptotic behavior for the logarithm of the number of k-solution-free collections in Abelian groups is obtained.**

***Keywords*— Set, characteristic function, group, progression, coset.**

I. Introduction

Let G be an Abelian group of order n, let $A_1, \dots, A_k$ be subsets of the group G, and let $k \geq 3$ be a natural number. A collection of sets $(A_1, \dots, A_k)$ is called a k-solution-free ((k, l)-SFC) if the equation

$$x_1 + \dots + x_k = 0 \tag{1}$$

has no solution in $(A_1, \dots, A_k)$, where $x_1 \in A_1, \dots, x_k \in A_k$. The family of all k-SFCs in G will be denoted by $S_k(G)$.

Let k and l be nonnegative integers such that $k + l \geq 3$. A subset $A \subseteq G$ is called a (k, l)-sum-free set ((k, l)-SFS) if the equation $x_1 + \ldots + x_k = y_1 + \ldots + y_l$ has no solutions in the set A. The family of all (k, l)-SFSs in G will be denoted by $SF_{k,l}(G)$. A $(2, 1)$-sum-free set is simply called a sum-free set (a SFS). Given natural numbers m and n, by $[m, n]$ we denote the set of all natural numbers x such that $m \leq x \leq n$. In 1988, Cameron and Erdös [1] conjectured that $SF_{2,1}([1, n]) = O(2^{n/2})$. They proved, in particular, that there exist constants c_0 and c_1 such that $|SF_{2,1}([[n/3], n])| \sim c_0 2^{n/2}$ for even n and $|SF_{2,1}([[n/3], n])| \sim c_1 2^{n/2}$ for odd n. Calkin [2] and independently Alon [3] showed that[1] $\limsup_{n\to\infty} \frac{2}{n} \log |SF_{2,1}([1, n])| \leq 1$. Sapozhenko [4] and independently Green [5] proved the Cameron-Erdös conjecture and found the asymptotic behavior of the number of SFSs in the interval $[1, n]$. In particular, it was shown that $|SF_{2,1}([1, n])| \sim c(n)2^{n/2}$, where the constant $c(n)$ depends on the parity of n. In 1991, Alon [3] showed that, for any $\varepsilon > 0$, the number of SFSs in an arbitrary finite group of order n is at most $2^{n/2+\varepsilon n}$ for all sufficiently large n. Later this result was refined for various subclasses of finite Abelian groups. In this way, in 2002, Sapozhenko [6] and independently Lev, Luczak, and Schoen [7] found the asymptotic behavior of the maximum possible number of SFSs for finite Abelian groups that contain at least one subgroup of index 2. By Z_n we shall denote a cyclic group of order n. In 2002, Lev and Schoen [8] showed that if p is a sufficiently large prime number, then

$$2^{\lfloor (p-2)/3 \rfloor}(p-1)(1 + O(2^{-\varepsilon_1 p})) \leq |SF_{2,1}(Z_p)| \leq 2^{p/2-\varepsilon_2 p},$$

where ε_1 and ε_2 are positive constants.

In 2005, Green and Ruzsa [9] used Fourier transforms to obtain asymptotics of the logarithm of the number of SFSs in finite Abelian groups. They showed that $\log |SF_{2,1}(G)| \sim \mu_{2,1}(G)$, for any finite Abelian group G, where $\mu_{2,1}(G)$ is the maximal cardinality of an SFS in G. In 2009, Sapozhenko [10] found the asymptotic behavior of the number of SFSs in groups of prime order.

At the same time, much effort has been devoted to extensions of the Cameron-Erdös problem. In particular, the number of (k, l)-SFSs was studied.

In 1996, Calkin and Taylor [11] proved that there exists a constant C_k, $k \geq 3$, such that $|SF_{k,1}([1, n])| \leq C_k 2^{(k-1)n/k}$. In 1998, Bilu [12] showed that $|SF_{l+1,l}([1, n])| = (1 + \bar{o}(1))2^{\lfloor (n+1)/2 \rfloor)}$, and Calkin and Thomson [13] established that $SF_{k,l}([1, n]) \leq C_{k,l} 2^{(k-l)n/k}$, for some constant $C_{k,l}$ with $k \geq 4l - 1$.

In 2000, Schoen [14] found the asymptotic behavior of the number of (k, l)-SFSs in the interval $[1, n]$ of natural numbers under some constraints on k and l. In 2003, Lev [15] estimated from above the number of (k, l)-SFSs in the interval $[1, n]$ of natural numbers. Sargsyan [16] found the asymptotic behavior of the logarithm of the number of (k, l)-SFSs for an arbitrary Abelian group. He showed that $|\log |SF_{k,l}(G)| - \mu_{k,l}(G)| < \varepsilon n$, for any $\varepsilon > 0$ all sufficiently large n, where $\mu_{k,l}(G)$ is the maximum cardinality of a (k, l)-SFS in G.

We set

$$\nu_k(G) = \max_{(A_1,\dots,A_k)\in S_k(G)} |A_1 \cup \dots \cup A_k|.$$

In the present paper we prove the following result.

Theorem 1: Let G be an Abelian group of order n and let $k \geq 3$ be a natural number. Then

$$\log |S_k(G)| = \nu_k(G) + \bar{o}(n),$$

as $n \to \infty$.

[1] Here and below, $\log x = \log_2 x$

II. Granulation

Definition 1: An L-granule of coset type is the union of cosets of a group G modulo some subgroup of order at least L.

Definition 2: Suppose that L is an integer, $d \in G$, and also $ord(d) \geq L$, where $ord(d)$ is the order of d. Let H be the subgroup in G generated by d. Partition each coset of H into $\lfloor ord(d)/L \rfloor$ progressions of the form $\{x + id \mid 0 \leq i \leq L-1\}$ and one "remainder" set of size less than L. For each $d \in G$, fix one of such partitions. The union of the so-obtained progressions (which does not include the "remainder" sets) is called an L-granule of progression type.

Remark 1: Note that, in the definition of an L-granule of coset (progression) type, the union of arbitrary cosets (progressions) is taken.

The following lemmas can be found in [[9], p. 166, Lemmas 3.3 and 3.4]:

Lemma 1: Suppose that n is a sufficiently large natural number, G is an Abelian group of order n, and $L \leq \sqrt{n}$. Then G contains at most $2^{3n/L}$ L-granules of both types (of progression and coset types).

Lemma 2: Suppose that n is a sufficiently large natural number, M is a set of size n, and ρ is a real number that is less than some absolute constant. Then the number of subsets in M of size at most ρn, is at most $2^{n\sqrt{\rho}}$.

The following is proved in [17]:

Theorem 2: Let $k \geq 3$ and let $A_1, \ldots, A_k$ be subsets in an Abelian group G of order n such that there exist $\overline{o}(n^{k-1})$ solutions to the equation $x_1 + \cdots + x_k = 0$ for $x_i \in A_i$, $i = 1, \ldots, k$. Then there exist subsets $A'_1, \ldots, A'_k$ such that $A'_i \subseteq A_i$, $|A_i \setminus A'_i| = \overline{o}(n)$, and there are no solutions to $x_1 + \cdots + x_k = 0$ for $x_i \in A'_i$.

*Lemma 3 (***Granulation***):* Let n be a sufficiently large natural number, G be an Abelian group of order n, and $k \geq 3$ be a natural number, $(A_1, \ldots, A_k) \in S_k(G)$, let $0 < \varepsilon < 1/2$, and let L, L' be two positive numbers satisfying the inequality

$$n > L' \left(4L/\varepsilon\right)^{k^2 4^{2k+1} \varepsilon^{-2(k+1)}}.$$

Then there exist subsets $A'_1, \ldots, A'_k \subseteq G$ such that

(i) $A'_1, \ldots, A'_k$ are either progression-type L-granules or coset-type L'-granules;

(ii) $\left|A_1 \setminus A'_1\right| \leq \varepsilon n, \ldots, \left|A_k \setminus A'_k\right| \leq \varepsilon n$;

(iii) $(A'_1, \ldots, A'_k)$ contains at most εn^{k-1} solutions of equation (1).

Proof: Without proof. □

III. The number of k-solution-free collections in an Abelian group

The following theorem proves the existence of a family of granules.

Theorem 3: Let G be an Abelian group of large order n.Then there exists a family $\mathcal{F}$ of collections $(F_1, \ldots, F_k)$ of subsets of the group G satisfying the following conditions:

(i) $\log |\mathcal{F}| \leq 2kn(k-1)^{-1/2}(\log n)^{-(4k+6)^{-1}}$;

(ii) for each $(A_1, \ldots, A_k) \in S_k(G)$ there exists a collection $(F_1, \ldots, F_k) \in \mathcal{F}$ such that $A_1 \subseteq F_1, \ldots, A_k \subseteq F_k$;

(iii) each collection $(F_1, \ldots, F_k) \in \mathcal{F}$ contains at most $n^{k-1}(\log n)^{-(2k+3)^{-1}}$ solutions of equation (1).

Proof: We set $L = L' = \lfloor \log n \rfloor$ and $\varepsilon = (k+1)^{-1}(\log n)^{-(2k+3)^{-1}}$. Note that, for a sufficiently large n, the hypotheses of Lemma 3 are satisfied for such parameters. So, for each collection $(A_1 \ldots, A_k) \in S_k(G)$, using Lemma 3 we construct a collection of sets $(A'_1 \ldots, A'_k)$. We set $\mathcal{F} = \{(A_1 \cup A'_1, \ldots, A_k \cup A'_k) \mid (A_1 \ldots, A_k) \in S_k(G)\}$. Hence assertion (ii) holds automatically. Therefore, from assertion (ii) of Lemma 3 it follows that the cardinality of the family $\mathcal{F}$ is majorized by the number of collections $(F_1 \ldots, F_k)$ such that, for all $i = 1, \ldots, k$, the set F_i is a union of an L-granule with some subset of the group G of cardinality at most εn. So, from Lemmas 1 and 2 it follows that $\log |\mathcal{F}| \leq k(3n/L + n\sqrt{\varepsilon})$. For sufficiently large n, this quantity is not greater than $2kn\sqrt{\varepsilon} = 2kn(k-1)^{-1/2}(\log n)^{-(4k+6)^{-1}}$. This proves assertion (i).

We next note that if an element is added to one of the sets from the collection $(F_1, \ldots, F_k) \in \mathcal{F}$, then in this set at most n^{k-2} new solutions of equation (1) can appear. Hence from assertion (iii) of Lemma 3 it follows that in each collection $(F_1, \ldots, F_k) \in \mathcal{F}$ there are at most $\varepsilon n^{k-1} + \varepsilon n \cdot kn^{k-2} = n^{k-1}(\log n)^{-(2k+3)^{-1}}$ solutions of equation (1). □

Theorem 4: Let G be an Abelian group of order n and let $k \geq 3$ be a natural number. Then

$$\log |S_k(G)| = \nu_k(G) + \overline{o}(n),$$

as $n \to \infty$.

Proof: By Theorem 3, there exists a family $\mathcal{F}$ of collections of subsets (granules) of the group G satisfying conditions (i)-(iii). Let $(F_1, \ldots, F_k) \in \mathcal{F}$. Given a fixed $(F_1, \ldots, F_k)$, it follows from Theorem 2 with $A_1 = F_1, \ldots, A_k = F_k$, that there exist $F'_1 \subseteq F_1, \ldots, F'_k \subseteq F_k$, such that $|F_i \setminus F'_i| = \overline{o}(n)$, $i = 1, \ldots, k$, and $(F'_1, \ldots, F'_k) \in S_k(G)$. An ordered collection $(Q_1, \ldots, Q_k)$ is called a "*subcollection*" of the collection $(W_1, \ldots, W_k)$, if $Q_1 \subseteq W_1, \ldots, Q_k \subseteq W_k$. Hence, since $\log |\mathcal{F}| = \overline{o}(n)$ (assertion (i) of Theorem 3), we see that the number of "*subcollections*" of all collections from the collection $\mathcal{F}$, is at most $2^{|F'_1 \cup \cdots \cup F'_k| + \overline{o}(n)}$.

By assertion (ii) of Theorem 3, each collection from $S_k(G)$ is a "*subcollection*" of some collection from the family $\mathcal{F}$. Therefore,

$$\log |S_k(G)| = \nu_k(G) + \overline{o}(n),$$

as $n \to \infty$. □

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