Statistical Accuracy of Nonlinear Circulant Control Systems

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***Abstract*—In the paper, the classical method of statistical linearization is extended to a special class of nonlinear multivariable control systems, called circulant systems. The extension is based on the characteristic transfer functions method which allows reducing the analysis of a linear (or linearized) cross-connected multivariable control system with N inputs and N outputs to N independent systems with one input and one output. It is shown that the analysis of statistical accuracy of nonlinear circulant systems is reduced to a one-dimensional task irrespective of the number of channels N. A numerical example is given.**

***Keywords*—Multivariable control system, nonlinear system, circulant system, characteristic transfer function, statistic linearization.**

I. Introduction

One of the central issues in modern feedback control is investigation of nonlinear systems affected by stochastic inputs or disturbances [1-5]. In the engineering practice, the most effective methods of analysis and design of such systems is based on the statistical (or stochastic) linearization which utilizes the probability distribution of the input signal of the nonlinearity to approximate the latter by a linear function consisting of an equivalent gain and a bias [1, 2].

In the paper, the classical method of statistical linearization for single-input single-output (SISO) nonlinear systems is extended to the case of a special class of nonlinear multi-input multi-output (MIMO) systems, called circulant systems. The extension of the statistical linearization method to circulant systems is based on the Characteristic Transfer Functions (CTFs) method which allows reducing the analysis of a linear (or linearized) cross-connected MIMO system with N inputs and N outputs to N independent SISO systems [6]. It is shown that due to the peculiar structural features of nonlinear circulant systems, the analysis of their statistical accuracy is reduced to a one-dimensional task irrespective of the number of channels N.

II. Nonlinear MIMO Systems

The matrix block diagram of a general N-dimensional (that is having N inputs and N outputs) nonlinear MIMO system is shown in Fig. 1 where: $\varphi(t)$, x, y are N-dimensional vectors; $W(s) = \{W_{kr}(s)\}$ is an $N \times N$ transfer matrix of the linear part; $F(x) = \{F_{kr}(x_r)\}$ is an $N \times N$ matrix of nonlinear elements depending on input variables x_r $(r = 1, 2, \ldots, N)$. For simplicity, we shall assume further that all nonlinearities $F_{kr}(x_r)$ are memoryless, odd and single-valued.

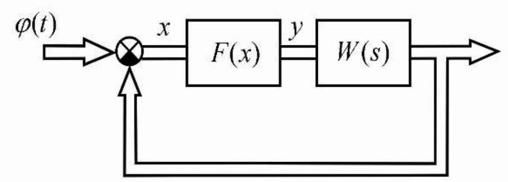

Fig. 1. Matrix block diagram of a nonlinear MIMO control system.

Let us suppose that the nonlinear MIMO system in Fig. 1 is stable (not self-oscillating), and the linear part $W(s)$ is of type m. Let the vector input signal $\varphi(t)$ be of the form

$$\varphi(t) = \varphi_0(t) + \gamma(t) , \tag{1}$$

where $\varphi_0(t)$ is a deterministic reference signal described by a polynomial of degree m with constant *vectors* v_k:

$$\varphi_0(t) = \sum_{k=1}^{m} \frac{v_k t^k}{k!} , \tag{2}$$

and $\gamma(t)$ is a stationary stochastic Gaussian process with zero means, which represents disturbances and is specified by a correlation matrix

$$R_\gamma(\tau) = E[\gamma(t)\gamma^T(t+\tau)] \tag{3}$$

or by a nonnegative-definite Hermitian matrix of power spectral densities $S_\gamma(j\omega)$.

In practice, the situation in which components of the input stochastic vector $\varphi(t)$ are uncorrelated and have identical power spectral densities $s_\gamma(\omega)$ is quite common [6]. Therefore, we shall assume further that the matrix $S_\gamma(j\omega)$ is scalar and equal to

$$S_\gamma(j\omega) = s_\gamma(\omega)I\,, \qquad (4)$$

where I is the unit matrix. The vector $x(t)$ in the steady-state condition should be approximately sought in the form

$$x(t) = x_0 + x_R(t)\,, \qquad (5)$$

where x_0 is a constant vector [the mathematical expectation of $x(t)$], and $x_R(t)$ is a centered stochastic function of time. Replace, in accordance with the statistical linearization method [1, 2], the real relationship between the input and the output of each nonlinear element $F_{ir}(x_r)$ by an *approximate* relationship

$$F_{ir}(x_r) = k_{ir}^0 x_{0r} + k_{1ir} x_{Rr}\,, \quad i, r = 1, 2, \ldots, N\,, \qquad (6)$$

where k_{ir}^0 and k_{1ir} are, respectively, the *statistical characteristic* and *statistical gain* of the nonlinearity $F_{ir}(x_r)$ [1, 2] with respect to the random component $x_R(t)$ of $x(t)$ (5). As shown in [7], the approximate analysis of statistical accuracy of the nonlinear MIMO system in Fig. 1 can be performed by a joint analysis of two cross-connected linear MIMO control systems in Fig. 2, in which the vector σ_x is composed of the standard deviations σ_{xr} ($r = 1,2,\ldots\ldots,N$) and the matrix $K_1(x_0, \sigma_x)$ is composed of the coefficients $k_{1ir}(x_{0r}, \sigma_{xr})$.

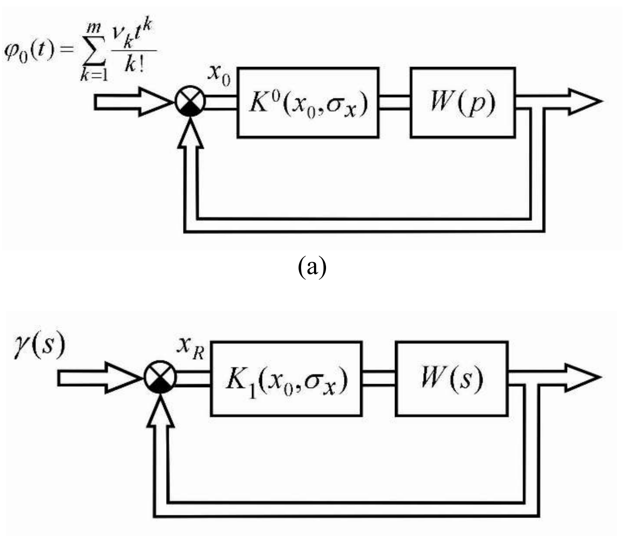

Fig. 2. Equivalent matrix block diagrams of the statistically linearized nonlinear MIMO system in Fig. 1. (a) Linearized system with respect to the constant vector x_0, (b) linearized system with respect to the random disturbance $\gamma(t)$.

III. NONLINEAR CIRCULANT SYSTEMS

Suppose now that the transfer matrix of the linear part $W(s)$ of the MIMO system in Fig. 1 is *circulant* [6], i.e. it can be represented as polynomial

$$W(s) = W_0(s)I + \sum_{i=1}^{N-1} W_i(s)U^i\,, \qquad (7)$$

where $W_0(s)$, $W_i(s)$ $(i = 1, 2, \ldots, N-1)$ are the elements of the first row of $W(s)$, and U is the orthogonal ($U^{-1} = U^T$) *permutation matrix* [8]

$$U = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \qquad (8)$$

Further, let the $F(x)$ matrix is also circulant by the structure of introducing the nonlinear elements into the system, i.e. it has the form

$$F(x) = \begin{pmatrix} F_0(x_1) & F_1(x_2) & F_2(x_3) & \ldots & F_{N-1}(x_N) \\ F_{N-1}(x_1) & F_0(x_2) & F_1(x_3) & \ldots & F_{N-2}(x_N) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ F_1(x_1) & F_2(x_2) & F_3(x_3) & \ldots & F_0(x_N) \end{pmatrix}. \qquad (9)$$

Then, instead of (9) we can, formally, write

$$F(x) = F_0(\cdot)I + \sum_{i=1}^{N-1} F_i(\cdot)U^i\,. \qquad (10)$$

It is important to note that the same linear and nonlinear elements are situated on each diagonal of $W(s)$ (7) and $F(x)$ (9), and, besides, the diagonals located at the same distance from the lower-left corner and from the principal diagonal consist of identical elements. This means that nonlinear circulant systems have internal structural symmetry. That ensures that in case of scalar matrix of power spectral densities $S_\gamma(j\omega)$ (4), all variances σ_{xr}^2 of the variables x_r are equal to each other, i.e. $\sigma_{xr}^2 = \sigma_x^2$.

Let us show that this significant feature really takes place, confining ourselves, for simplicity, to the case of circulant systems with odd-symmetrical nonlinearities and assuming that the mathematical expectation of the input stochastic signal $\varphi(t) = \varphi_0(t) + \gamma(t)$ is zero [$\varphi_0(t) = 0$].

Suppose that, for $S_\gamma(j\omega) = s_\gamma(\omega)I$, all variances of the input variables x_r of the nonlinear elements are equal, i.e. $\sigma_{xr}^2 = \sigma_x^2$ ($r = 1, 2, \ldots, N$). In this case, the matrix $K_1(\sigma_x)$ composed of the statistical gains $k_{1i}(\sigma_x)$ $(i = 0, 1, \ldots, N-1)$ of the elements of the structurally circulant matrix $F(x)$ (10) is a really circulant matrix, for which we can write down

$$K_1(\sigma_x) = k_{10}(\sigma_x)I + \sum_{i=1}^{N-1} k_{1i}(\sigma_x)U^i\,. \qquad (11)$$

As a result, the transfer function matrices

$$Q(s, \sigma_x) = W(s)K_1(\sigma_x) \qquad (12)$$

and

$$\Phi_R(s,\sigma_x) = [I + Q(s,\sigma_x)]^{-1} \quad (13)$$

of the open- and closed-loop statistically linearized circulant system are also circulant [note that the matrices $Q(s,\sigma_x)$ and $\Phi_R(s,\sigma_x)$ in equations (12) and (13) depend on *one* scalar unknown σ_x]. For the transfer matrix $Q(s,\sigma_x)$, denoting the elements of the first row by $Q_0(s,\sigma_x)$, $Q_i(s,\sigma_x)$ $(i = 1, 2, ..., N-1)$, we have

$$Q(s,\sigma_x) = Q_0(s,\sigma_x)I + \sum_{i=1}^{N-1} Q_i(s,\sigma_x)U^i . \quad (14)$$

The CTFs $q_i(s,\sigma_x)$ of the matrix $Q(s,\sigma_x)$ can be written in analytical form for any number of channels N:

$$q_i(s,\sigma_x) = Q_0(s,\sigma_x) + \sum_{k=1}^{N-1} Q_k(s,\sigma_x)\exp\left\{ j\frac{2\pi(i-1)}{N}k \right\}$$

$$i = 1, 2, ..., N. \quad (15)$$

Under the above assumptions, the matrix $S_x(j\omega)$ of power spectral densities of the vector x:

$$S_x(j\omega) = s_\gamma(\omega)\,\Phi_x(j\omega,\sigma_x)\,\Phi_x^*(j\omega,\sigma_x) \quad (16)$$

is also circulant, i.e. each subsequent row of $S_x(j\omega)$ is obtained from the preceding row by shifting all elements (except for the N th) by one position to the right; the N th element of the preceding row then becomes the first element of the following row. Besides, being the matrix of spectral densities, the matrix $S_x(j\omega)$ belongs to the class of nonnegative-definite Hermitian matrices and coincides with its conjugate matrix.

Recall that the eigenvalues of the Hermitian circulant matrix, like any other Hermitian matrix, are always real-valued, and the modal matrix is unitary and is inherited (due to being circulant) from the permutation matrix U (8). Recall also that the diagonal elements of any Hermitian circulant matrix are real-valued (as the matrix is Hermitian) and equal to each other (as the matrix is circulant). From here, we immediately arrive at a conclusion that the diagonal elements $s_{xrr}(\omega)$ of the matrix of power spectral densities $S_x(j\omega)$ (16) are equal to each other. Consequently, the integrals of the equal diagonal elements $s_{xrr}(\omega) = s_x(\omega)$ ($r = 1, 2, ..., N$) taken over infinite limits from $-\infty$ to $+\infty$, which give by definition the variances σ_{xr}^2 of the variables x_r, coincide, i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} s_x(\omega)\,d\omega = \sigma_{xr}^2 = \sigma_x^2 \quad (17)$$

$$r = 1, 2, ..., N$$

which was to be proved.

Taking into account equations (12)-(15) and the equality $\sigma_{xr}^2 = \sigma_x^2$, the transfer matrix $Q(s,\sigma_x)$ (12) and $\Phi_R(s,\sigma_x)$ (13) of the open- and closed-loop statistically linearized circulant system can be represented in the canonical form

$$Q(s,\sigma_x) = C\,diag\{q_i(s,\sigma_x)\}\,C^{-1} \quad (18)$$

$$\Phi_R(s,\sigma_x) = C\,diag\left\{\frac{1}{1 + q_i(s,\sigma_x)}\right\}C^{-1}, \quad (19)$$

where C is a *constant* unitary modal matrix composed of the normalized eigenvectors of the permutation matrix U (8) [7]. Since both matrices $W(s)$ and $K_1(\sigma_x)$ in (12) are circulant, the CTFs $q_i(s,\sigma_x)$ of the open-loop system can be written as a product of two CTFs $q_i^L(s)$ and $q_i^N(\sigma_x)$ of the 'linear' $W(s)$ and 'nonlinear' $K_1(\sigma_x)$ matrices:

$$q_i(s,\sigma_x) = q_i^L(s)\,q_i^N(\sigma_x), \quad i = 1, 2, ..., N, \quad (20)$$

where

$$q_i^L(s) = W_0(s) + \sum_{k=1}^{N-1} W_k(s)\exp\left\{ j\frac{2\pi(i-1)}{N}k \right\} \quad (21)$$

$$q_i^N(\sigma_x) = k_{10}(\sigma_x) + \sum_{k=1}^{N-1} k_{1k}(\sigma_x)\exp\left\{ j\frac{2\pi(i-1)}{N}k \right\}. \quad (22)$$

The corresponding block diagram of one-dimensional characteristic systems for the statistically linearized circulant system takes on the form depicted in Fig. 3.

Using the canonical representation of the transfer matrix $\Phi_R(s,\sigma_x)$ (19), substituting in (19) $s = j\omega$ and taking into account that for the unitary modal matrix C, the relation $C^{-1} = C^*$ holds, we obtain from (16):

$$S_x(j\omega,\sigma_x) = C\,diag\left\{\frac{s_\gamma(\omega)}{|1 + q_i(j\omega,\sigma_x)|^2}\right\}C^{-1}. \quad (23)$$

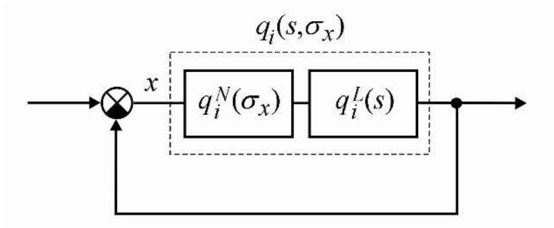

Fig. 3. Block diagram of the closed-loop characteristic systems of the statistically linearized circulant system.

From this equation, we conclude that the canonical basis of the positive-definite Hermitian matrix $S_x(j\omega,\sigma_x)$ coincides with the canonical basis of the circulant system, and the eigenvalues $\beta_{xi}(j\omega,\sigma_x)$ of the matrix $S_x(j\omega,\sigma_x)$ have the form:

$$\beta_{xi}(j\omega,\sigma_x) = \frac{1}{|1 + q_i(j\omega,\sigma_x)|^2}s_\gamma(\omega) \quad (24)$$

$$i = 1, 2, ..., N.$$

Integrating both parts in equation (24) over the frequency ω between the infinite limits $-\infty$ and $+\infty$, and allowing for (16) and the following standard expression for the covariance matrix P_x of the vector x [6]:

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_x(j\omega, x_0, \sigma_x)\, S_\gamma(j\omega)\, \Phi_x^*(j\omega, x_0, \sigma_x)\, d\omega, \quad (25)$$

yields the canonical representation of the covariance matrix P_x:

$$P_x = C\, diag\{D_i(\sigma_x)\}\, C^{-1}, \quad (26)$$

where $D_i(\sigma_x)$ are the variances of the "errors" of the characteristic systems defined by the formulae

$$D_i(\sigma_x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{|1+q_i(j\omega,\sigma_x)|^2}\, s_\gamma(\omega)\, d\omega \quad (27)$$
$$i = 1, 2, \ldots, N.$$

Thus, if all input signals of a stable nonlinear circulant system are centered random processes described by the *scalar* matrix of power spectral densities $S_\gamma(j\omega)$ (4), then the covariance matrix P_x has identical diagonal elements and is connected with the diagonal matrix $diag\{D_i(\sigma_x)\}$ of the variances $D_i(\sigma_x)$ of the SISO characteristic systems via the similarity transformation (26).

From the theory of matrices [8], it is known that the trace (the sum of diagonal elements) of a matrix is invariant to the similarity transformation. Consequently, the trace $tr\{P_x\}$ of the covariance matrix P_x, which is equal in our case to $N \cdot \sigma_x^2$, at the same time is equal to the sum of the variances $D_i(\sigma_x)$. From here, we finally obtain the formula

$$\sigma_x^2 = \frac{1}{2\pi N}\sum_{i=1}^{N}\left\{\int_{-\infty}^{+\infty} \frac{s_\gamma(\omega)}{|1+q_i(j\omega,\sigma_x)|^2}\, d\omega\right\}. \quad (28)$$

Thus, the identical [for $S_\gamma(j\omega) = s_\gamma(\omega) I$] variances σ_x^2 at the inputs of nonlinearities of the circulant system of an arbitrary dimension N are equal to the arithmetic mean value of the 'error' variances $D_i(\sigma_x)$ of the characteristic systems, and can be found by solving a nonlinear *scalar* equation. The only unknown σ_x in (28) can be found by means of classical techniques described, for example, in [1].

In other words, the problem of determining the variances σ_x^2 in the nonlinear statistically linearized circulant system with any number of channels N proves to be, in principle, no more complicated than in the common classical (SISO) case – only the numerical computations become more intensive.

IV. Numerical Example

Analyze, on the basis of the above technique, the statistical accuracy of the nonlinear three-dimensional circulant system, the matrix block diagram of which is shown in Fig. 4, where the saturation nonlinearities in the separate channels (Fig. 5) have unity gains and linear zones equal to $\Delta = \pm 3$ [the matrix $F(x)$ is assumed to be diagonal].

The parameters K and T of the identical transfer functions of the system channels are equal to $K = 10$, $T = 0.2$ s.

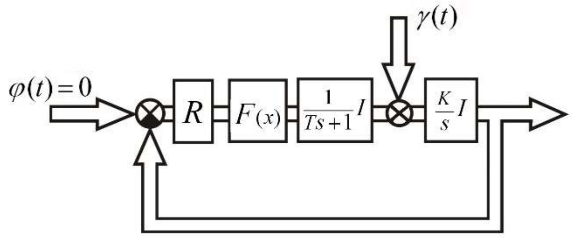

Fig. 4. Block diagram of the nonlinear three-dimensional circulant system.

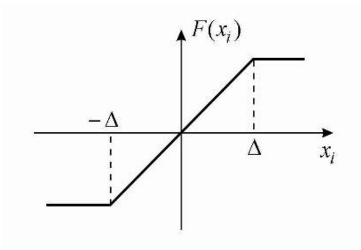

Fig. 5. Saturation nonlinearity.

The numerical matrix of the system cross-connections R has the form

$$R = \begin{pmatrix} \cos\varepsilon & \frac{1}{\sqrt{3}}\sin\varepsilon & -\frac{1}{\sqrt{3}}\sin\varepsilon \\ -\frac{1}{\sqrt{3}}\sin\varepsilon & \cos\varepsilon & \frac{1}{\sqrt{3}}\sin\varepsilon \\ \frac{1}{\sqrt{3}}\sin\varepsilon & -\frac{1}{\sqrt{3}}\sin\varepsilon & \cos\varepsilon \end{pmatrix}. \quad (29)$$

It can be shown that the eigenvalues of the matrix R (29) have the form

$$\lambda_1 = \cos\varepsilon$$
$$\lambda_{2,3} = \cos\varepsilon \pm j\sin\varepsilon = \exp\{\pm j\varepsilon\}, \quad (30)$$

and for values of the angle ε less than $\varepsilon_k = 38.68^\circ$, which is equal to the phase margin of the isolated separate channels of the system, the nonlinear system of Fig. 4 has no self-oscillation, i.e. the system is stable.

Let the components of the exogenous disturbance $\gamma(t)$ be uncorrelated stationary Gaussian processes with zero mathematical expectations having constant power spectral density N_0 (white noise). Under these assumptions, the identical variances σ_x^2 of the input variables of nonlinearities are determined by (28). The transfer matrix of the closed-loop statistically linearized system $\Phi_R(s,\sigma_x)$, which relates the vector x with the disturbance $\gamma(t)$ is

$$\Phi_R(s,\sigma_x) = -\left[I + \frac{10k_1(\sigma_x)}{s(0.2s+1)}R\right]^{-1}, \quad (31)$$

or, in the canonical form,

$$\Phi_R(s,\sigma_x) = -C\, diag\left\{\frac{10\lambda_i(0.2s+1)}{0.2s^2 + s + 10\lambda_i k_1(\sigma_x)}\right\}C^{-1}, \quad (32)$$

where the statistical gain $k_1(\sigma_x)$ is given by the known expression [1]. From equation (32), it can be seen that the transfer functions $\Phi_{Ri}(s,\sigma_x)$ of the closed-loop characteristic systems are equal, up to the minus sign, to

$$\Phi_{Ri}(s,\sigma_x) = \frac{10\lambda_i(0.2s+1)}{0.2s^2+s+10\lambda_i k_1(\sigma_x)}, \quad i=1,2,3. \tag{33}$$

To determine the variances $D_i(\sigma_x)$ of one-dimensional characteristic systems, the ready formulae given in [6] can be used, from which we obtain the following analytical expressions:

$$D_i(\sigma_x) = \frac{1}{2}\frac{|\lambda_i|^2[1+\mathrm{Re}\{\lambda_i\}Kk_1(\sigma_x)T]N_0K}{k_1(\sigma_x)[\mathrm{Re}\{\lambda_i\}-(\mathrm{Im}\{\lambda_i\})^2Kk_1(\sigma_x)T]} \tag{34}$$

Using equations (28) and (34) yields the following scalar nonlinear equation in the unknown standard deviation σ_x:

$$\sigma_x^2 = \frac{5N_0\left[1+2k_1(\sigma_x)\cos\varepsilon\right]}{3k_1(\sigma_x)}\left[\cos\varepsilon + \frac{2}{\cos\varepsilon - 2k_1(\sigma_x)(\sin\varepsilon)^2}\right] \tag{35}$$

The results of the numerical evaluation of equation (35) for different values of the angle $\varepsilon = const$ ($\varepsilon = 0^\circ, 20^\circ, 30^\circ$), as the intensity of the white noise N_0 changes from 0 to 3, are given in graphical form in Fig. 6.

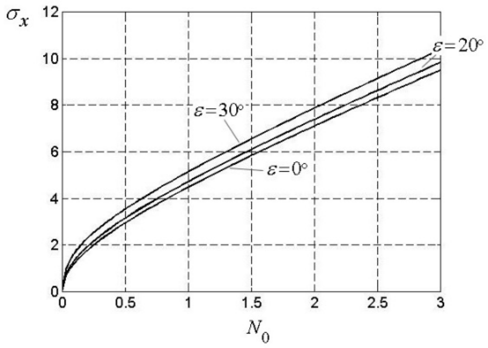

Fig. 6. The results of the solution of equation (35).

For $\varepsilon = 0^\circ$, the matrix R (29) passes into the unit matrix I and equation (35) takes on the simple form

$$\sigma_x = \sqrt{\frac{5N_0\left[1+2k_1(\sigma_x)\right]}{k_1(\sigma_x)}}, \tag{36}$$

which defines the standard deviation σ_x of isolated separate channels of the system.

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