On the Probabilistic Model of the Cartesian Product of Canonically Conjugated Fuzzy Subsets

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***Abstract*—In the present work the problems related to the construction of a probabilistic model of the Cartesian product of canonically conjugated fuzzy subsets is investigated. The case of the Cartesian product of two fuzzy subsets with commuting colors is considered in detail. It is shown that the model most fully reflects the special "additional" nature of the connection between two canonically conjugated colors.**

***Keywords*—Probabilistic model, canonically conjugated fuzzy subsets, color operators.**

I. Introduction

The main content of this work is the problem of constructing a probabilistic model of the Cartesian product of canonically conjugated fuzzy subsets, in which the compatibility function is calculated on the basis of the corresponding characteristic function of the Cartesian product. For the conciseness and simplicity of presentation, we will use the definitions and terminology introduced in [1-3].

In the theory of canonically conjugated fuzzy subsets, it is usually assumed that canonically conjugated colors deal with the properties of information functions and linear operators in Hilbert space so that each information state corresponds to the assessment of the compatibility function, and each color - to the operator. In this case, the information state means a set of information that is a result of the activity of the expert making the measurements.

In the first section of this work, the characteristic function of the Cartesian product of two canonically conjugated fuzzy subsets is introduced into the consideration, on the basis of which, in the case of commuting color operators, relationships are obtained to calculate the density of the distribution of the Cartesian product of colors. It is also shown here that these expressions can be easily generalized to the case of the Cartesian product of any finite number of commuting color multipliers.

In the second section, it is shown that from the probabilistic model of the Cartesian product of two fuzzy subsets, the expression can be obtained, which establishes a connection between the dispersions of canonically conjugated colors.

II. Cartesian Product of Colors

The construction of a probabilistic model of a Cartesian product of canonically conjugated fuzzy subsets is of particular interest, since the definition of a Cartesian product given here differs from the one proposed by Zade [4]. It's important to note that this model fully reflects the special ("additional") nature of the connection between two canonically conjugated colors $\wp$ and $\wp^c$.

Before the construction of a Cartesian model, we will consider some basic definitions and concepts from [1] regarding the numerical characteristic of the color of $\wp$, which will be used in the future.

Assumption (basic). The numerical characteristic $\xi_\wp$ of color $\wp$ is a random variable ($\xi_\wp \in \mathbb{R}$), the probability distribution of which is characterized by density $\rho_\wp(x_\omega)$,

$$\int_R \rho_\wp(x_\omega)\,dx_\omega = 1.$$

Definition 1. A quantity

$$x^*_\omega \equiv M\xi_\wp = \int_R x_\omega \rho_\wp(x_\omega)\,dx_\omega,$$

is a mathematical expectation, called a computable value of color $\wp$ at the point of the universal set Ω.

$$\sigma^2_\wp(\omega) \equiv D\xi_\wp = \int_R (x_\omega - x^*_\omega)\rho_\wp(x_\omega)\,dx_\omega.$$

Note that the formula for x^*_ω establishes the relation between the set $\mathbb{R}$ of computable values of color $\wp$ and the universal set Ω.

The presence of color $\wp$ at the point ω, in addition to the value $M\xi_\wp$ is characterized by dispersion:

$$\sigma^2_\wp(\omega) \equiv D\xi_\wp = \int_R (x_\omega - x^*_\omega)\rho_\wp(x_\omega)dx_\omega.$$

It is important to note that namely $\sigma^2_\wp(\omega)$ is associated with the uncertainty of the color value $\wp$ at the point ω. If $\sigma^2_\wp(\omega) \to 0$, then we can say that the color $\wp$ at the point ω has a well-defined value. The larger the variance $\sigma^2_\wp(\omega)$, the more uncertain the color $\wp$ at the point ω. If $\sigma^2_\wp(\omega) \to \infty$, then ω does not have the color $\wp$.

Definition 2. The information function of color $\wp$ is called the expression

$$|x_\omega;\ \wp\rangle = \sqrt{\rho_\wp(x_\omega)}e^{i\varphi_\wp},$$

where the real value φ is an arbitrary phase**.**

Remark. This function, that is, Dirac's parentheses [5], we will use to represent the information (uncertainties) contained in the color $\wp$. The square of the information function module represents the compatibility function, more precisely, the corresponding density

$$\rho_\wp(x_\omega) = |x_\omega;\ \wp\rangle^+|x_\omega;\ \wp\rangle = \langle x_\omega;\wp|x_\omega;\ \wp\rangle.$$

When calculating the compatibility function of the Cartesian product of two fuzzy subsets, we will proceed from the corresponding characteristic function.

Let be given two fuzzy subsets of R_1 and R_2 , $\wp_1$ and $\wp_2$ are the colors, using which we can differ the elements of two subsets, and $\hat{\wp}_1$ and $\hat{\wp}_2$ are the corresponding operators.

At first, we will consider the case of commuting operators.

Using Dirac parentheses, we denote through $|x_{\omega_1}, x_{\omega_2};\ \wp_1, \wp_2\rangle$ the function that belongs to $L^2(R_1 \times R_2)$ (Hilbert space) and determines the compatibility function of the Cartesian product:

$$\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2}) =$$
$$= \langle x_{\omega_1}, x_{\omega_2};\wp_1,\wp_2|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle. \quad (1)$$
$$\mu_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2}) = \quad (2)$$

$$\iint_Q \langle x_{\omega_1}, x_{\omega_2};\wp_1,\wp_2|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle dx_{\omega_1}dx_{\omega_2},$$

where

$$Q = \Pi_{\wp_1\times\wp_2}(\omega_1,\omega_2) = \Pi_{\wp_1}(\omega_1)\times\Pi_{\wp_2}(\omega_2). \quad (3)$$

Definition 3. Two fuzzy numbers $x^*_{\omega_1}$ and $x^*_{\omega_2}$ are called non-interacting if

$$|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle = |x_{\omega_1};\ \wp_1\rangle\cdot|x_{\omega_2};\ \wp_2\rangle. \quad (4)$$

Otherwise, these numbers will be called interacting numbers.

Consider the operator

$$\hat{M}(\alpha_1,\alpha_2) = exp[i(\alpha_1\hat{\wp}_1 + \alpha_2\hat{\wp}_2)]. \quad (5)$$

Note that if $\hat{\wp}_1$ and $\hat{\wp}_2$ commute, then the operator (5) has a completely obvious meaning.

Definition 4. The scalar product

$$M(\alpha_1,\alpha_2) = \quad (6)$$
$$= \left(\langle x_{\omega_1}, x_{\omega_2};\wp_1,\wp_2|\,\hat{M}(\alpha_1,\alpha_2)|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle\right)$$

is called the characteristic function of color $\wp_1 \times \wp_2$.

Theorem. The density $\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2})$ is calculated using the formula:

$$\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2}) = \frac{1}{4\pi^2}\cdot \quad (7)$$

$$\cdot \int_{R_1}\int_{R_2} M(\alpha_1,\alpha_2)exp[-i(\alpha_1 x_{\omega_1} + \alpha_2 x_{\omega_2})]d\alpha_1 d\alpha_2.$$

Proof: Substitute (6) in (7):

$$\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2}) = \quad (8)$$

$$= \frac{1}{4\pi^2}\iint_{R_1\times R_2} d\alpha_1 d\alpha_2 exp[-i(\alpha_1 x_{\omega_1} + \alpha_2 x_{\omega_2})] =$$

$$= \iint_{R_1\times R_2} dx'_{\omega_1}dx'_{\omega_2}\cdot$$

$$\cdot\langle x'_{\omega_1}, x'_{\omega_2};\ \wp_1,\wp_2|exp[i(\alpha_1\hat{\wp}_1 + \alpha_2\hat{\wp}_2)]|x'_{\omega_1}, x'_{\omega_2};\ \wp_1,\wp_2\rangle.$$

When $x^*_{\omega_1}$ and $x^*_{\omega_2}$ are non-interacting fuzzy numbers, then it is obvious that

$$\hat{\wp}_i|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle = x_i|x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle, \quad (9)$$
$$i = 1,2.$$

It is natural to assume that the same relations occur in the case of interacting fuzzy numbers. So, we can write:

$$\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2}) =$$

$$= \iint_{R_1\times R_2} \left|\langle x'_{\omega_1}, x'_{\omega_2};\ \wp_1,\wp_2\rangle\right|^2 \delta(x'_{\omega_1} - x_{\omega_1})\cdot$$

$$\cdot\,\delta(x'_{\omega_2} - x_{\omega_2})dx'_{\omega_1}dx'_{\omega_2} = \left|\langle x_{\omega_1}, x_{\omega_2};\ \wp_1,\wp_2\rangle\right|^2,$$

where δ is a generalized Kronecker symbol.

It should be noted that using the density of the scalar product $\rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2})$, you can calculate distribution densities for colors $\wp_1$ and $\wp_2$:

$$\rho_{\wp_1}(x_{\omega_1}) = \int_R \rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2})dx_{\omega_2}, \quad (10)$$

$$\rho_{\wp_2}(x_{\omega_2}) = \int_R \rho_{\wp_1\times\wp_2}(x_{\omega_1}, x_{\omega_2})dx_{\omega_1}. \quad (11)$$

The ratios (5) to (11) can be easily generalized to the case of the Cartesian product of any finite number of commuting color multipliers:

$$\widehat{M}(\alpha_1, \cdots, \alpha_n) = exp(i\sum_{k=1}^{n}\alpha_k\widehat{\wp}_k), \quad (12)$$

$$M(\alpha_1, \cdots, \alpha_n) = \quad (13)$$

$$\langle x_{\omega_1}, .., x_{\omega_n}; \wp_1, .., \wp_n | \widehat{M}(\alpha_1, .., \alpha_n) | x_{\omega_1}, .., x_{\omega_n}; \wp_1, .., \wp_n \rangle$$

$$\rho_{\wp_1\times\cdots\times\wp_n}(x_{\omega_1}, \cdots, x_{\omega_n}) = \quad (14)$$

$$= \frac{1}{(2\pi)^n}\int_{R^n} M(\alpha_1, \cdots, \alpha_n)e^{-i\sum_{k=1}^{n}\alpha_k x_{\omega_k}}\prod_{k=1}^{n} d\alpha_k .$$

Note that with this density of commuting colors it is possible to obtain densities corresponding to a smaller number of multipliers by integration over the corresponding variables.

The various formulations of problems are possible in which information functions in phase space can be associated with both the information state and the colors evaluated (observed) by experts. But the case of the Cartesian product of two fuzzy subsets with non-commuting colors requires special consideration. Recall [1] that such a formalism is based on the well-known Weyl transformation [6] and the use of Wigner functions [7].

III. Relationship Between Variances Of Canonically Conjugated Colors

Now we will show that from the probabilistic model of the Cartesian product of two fuzzy subsets the relation

$$\sigma_{\wp}^2 \ \sigma_{\wp^c}^2 \geq \frac{c^2}{4} \quad (15)$$

follows for dispersions $\sigma_{\wp}^2$ and $\sigma_{\wp^c}^2$ of canonically conjugated colors $\wp$ and $\wp^c$ in the form of the uncertainty principle [1].

This fact indicates the advantage of the proposed model of Cartesian product.

We denote using $\hat{\alpha}$ and $\hat{\beta}$ the operators with zero averages

$$\langle x_\omega; \wp | \hat{\alpha} | x_\omega; \wp \rangle = \langle x_\omega; \wp | \hat{\beta} | x_\omega; \wp \rangle.$$

Using the well-known Schwartz inequality, we get

$$\langle x_\omega; \wp | \hat{\alpha}\hat{\beta} | x_\omega; \wp \rangle \leq \quad (16)$$

$$\leq (\langle x_\omega; \wp | \hat{\alpha}^2 | x_\omega; \wp \rangle)^{\frac{1}{2}} (\langle x_\omega; \wp | \hat{\beta}^2 | x_\omega; \wp \rangle)^{\frac{1}{2}}.$$

The operators $\hat{\alpha}$ and $\hat{\beta}$ can also be matched with random variables $\alpha(x_\omega)$ and $\beta(x_\omega)$ so that the following relations are performed

$$\langle x_\omega; \wp | \hat{\alpha}\hat{\beta} | x_\omega; \wp \rangle =$$

$$= \int_R \alpha(x_\omega)\beta(x_\omega)\rho_\wp(x_\omega)dx_\omega \equiv \overline{\alpha\beta}, \quad (17)$$

$$\langle x_\omega; \wp | \hat{\alpha}^2 | x_\omega; \wp \rangle =$$

$$= \int_R \alpha^2(x_\omega)\rho_\wp(x_\omega)dx_\omega \equiv \overline{\alpha^2} = \sigma_\alpha^2, \quad (18)$$

$$\langle x_\omega; \wp | \hat{\beta}^2 | x_\omega; \wp \rangle =$$

$$= \int_R \beta^2(x_\omega)\rho_\wp(x_\omega)dx_\omega \equiv \overline{\beta^2} = \sigma_\beta^2. \quad (19)$$

Therefore, if we put $\alpha(x_\omega) = (x_{c\omega})^*_{x_\omega} - (x^*_{c\omega})^*_{x_\omega}$ and $\beta(x_\omega) = x_\omega - x^*_\omega$, where

$$(x^*_{c\omega})^*_{x_\omega} = \int_R (x_{c\omega})^*_{x_\omega}\rho_\wp(x_\omega)dx_\omega , \quad (20)$$

then

$$\left|\int_R ((x_{c\omega})^*_{x_\omega} - (x^*_{c\omega})^*_{x_\omega})(x_\omega - x^*_\omega)\rho_\wp(x_\omega)dx_\omega\right| \leq$$

$$\leq \sigma_\alpha\sigma_\beta. \quad (21)$$

Next, consider a random variable

$$\alpha'(x_\omega) = \frac{d}{dx_\omega}\ln\rho_\wp(x_\omega) = \frac{1}{\rho_\wp(x_\omega)}\frac{d}{dx_\omega}\rho_\wp(x_\omega). \quad (22)$$

It is evident, that

$$\overline{\alpha'(x_\omega)} = \int_R \left(\frac{d}{dx_\omega}\ln\rho_\wp(x_\omega)\right)\rho_\wp(x_\omega)dx_\omega =$$

$$= \int_R \rho'_\wp(x_\omega)dx_\omega = \rho_\wp(x_\omega)\Big|_{-\infty}^{+\infty} = 0, \quad (23)$$

$$\overline{\alpha'^2(x_\omega)} = \int_R \left(\frac{d}{dx_\omega}\ln\rho_\wp(x_\omega)\right)^2\rho_\wp(x_\omega)dx_\omega =$$

$$= \int_R \rho_\wp^{-2}(x_\omega)\rho'^2_\wp(x_\omega)\rho_\wp(x_\omega)dx_\omega.$$

Suppose that $\frac{d}{dx_\omega}|x_\omega; \wp\rangle \in L^2(R)$. In this case

$$\int_R \frac{d^2}{dx_\omega^2}\ln\rho_\wp(x_\omega)\rho_\wp(x_\omega)dx_\omega =$$

$$= \int_R \frac{d}{dx_\omega}\left(\frac{\rho'_\wp(x_\omega)}{\rho_\wp(x_\omega)}\right)\rho_\wp(x_\omega)dx_\omega =$$

$$= \int_R \rho''_\wp(x_\omega)dx_\omega - \int_R \left(\frac{\rho'_\wp(x_\omega)}{\rho_\wp(x_\omega)}\right)^2\rho_\wp(x_\omega)dx_\omega$$

$$= -\int_R \left(\frac{\rho'_\wp(x_\omega)}{\rho_\wp(x_\omega)}\right)^2\rho_\wp(x_\omega)dx_\omega .$$

Therefore

$$\overline{\alpha'^2(x_\omega)} = -\int_R \left(\frac{d^2}{dx_\omega^2}\ln\rho_\wp(x_\omega)\right)\rho_\wp(x_\omega)dx_\omega .$$

Considering further the formula for the conditional dispersion $\sigma^2_{\wp^c}$ of color $\wp^c$ from [1],

$$\sigma^2_{\wp^c} = (x^2_{c\omega})^* - ((x_\omega)^*)^2 = -\frac{c^2}{4}\frac{d^2}{dx^2_{c\omega}}\ln\rho_{\wp}(x_\omega), \tag{24}$$

we will get:

$$\overline{\alpha'^2(x_\omega)} = \frac{4}{c^2}\int_R \sigma^2_{\wp^c}\langle x^*_{c\omega}|x_\omega\rangle\,\rho_{\wp}(x_\omega)dx_\omega\,, \tag{25}$$

$$\overline{\alpha'(x_\omega)(x_\omega - x^*_\omega)} =$$

$$= \int_R (x_\omega - x^*_\omega)\left(\frac{d}{dx_{c\omega}}\ln\rho_{\wp}(x_\omega)\right)\rho_{\wp}(x_\omega)dx_\omega =$$

$$= \int_R (x_\omega - x^*_\omega)\frac{d}{dx_{c\omega}}\rho_{\wp}(x_\omega)dx_\omega = \tag{26}$$

$$(x_\omega - x^*_\omega)\rho_{\wp}(x_\omega)\Big|^{+\infty}_{-\infty} - \int_R \rho_{\wp}(x_\omega)dx_\omega = -1.$$

By comparing (24), (25) and the definition of $\alpha(x_\omega)$, we see that we can put $\alpha'(x_\omega) = \alpha(x_\omega)$. Therefore (21) gives:

$$\sigma^2_{\wp}\ \int_R\ \sigma^2_{\wp^c}\ \langle x^*_{c\omega}|x_\omega\rangle\rho_{\wp}(x_\omega)dx_\omega\ \geq\frac{c^2}{4}.$$

Since the integral in this inequality is $\sigma^2_{\wp^c}$, we get the ratio (15).

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