On Initial Segments of the Class of Turing Degrees Containing Hypersimple *T*-Mitotic but not *wtt*-Mitotic Sets

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***Abstract*—We consider the properties of computably enumerable (c.e.) Turing degrees containing sets, which possess the property of a *T*-mitotic splitting but don't have a *wtt*-mitotic splitting.**

It is proved that for any noncomputable c.e. degree *b* there exists a degree *a*, such that *a* ≤ *b* and *a* contains a hypersimple *T*-mitotic set, which is not *wtt*-mitotic.

***Keywords*— Mitotic set, *T*-reducibility, *wtt*-reducibility, Hypersimple set, Contiguous degree.**

I. Introduction

Note that the elements of the class mentioned in the title are degrees containing hypersimple sets, which are simultaneously *T*-mitotic and non-*wtt*-mitotic.

Now we give the necessary definitions.

We shall use the notions and terminology introduced in Soare [1], Rogers [2].

Notations. We deal with sets and functions over the nonnegative integers $\omega = \{0,1,2,\ldots\}$.

Let $\omega_{ev} = \{x : (\exists k)(x = 2k)\}$; $\omega_{od} = \{x : (\exists k)(x = 2k+1)\}$.

Let φ_e be the e^{th} partial computable function in the standard listing (Soare [1], p.15, p.25).

If $A \subseteq \omega$ and $e \in \omega$, let $\Phi_e^A(x) = \Phi_e(A : x) = \{e\}^A(x)$ (see Soare [1], pp. 48-50).

χ_A denotes the characteristic function of A, which is often identified with A and written simply as $A(x)$.

Let $\varphi(x)\downarrow$ denote that $\varphi(x)$ is defined, $\varphi(x)\uparrow$ denote that $\varphi(x)$ is undefined.

$W_e = dom\,\varphi_e = \{x : \varphi_e(x)\downarrow\}$.

$\varphi_{e,at\,s+1}(x)\downarrow$ denotes $\varphi_{e,s+1}(x)\downarrow$ & $\varphi_{e,s}(x)\uparrow$.

$x \in W_{e,at\,s+1}$ denotes $x \in W_{e,s+1} - W_{e,s}$

(see the definitions of $\varphi_{e,s}$ and $W_{e,s}$ in Soare [1], p. 34).

In the literature, Turing reducibility is usually taken as the main reducibility. If the word "reducibility" is used without a further explanation, it means, as a rule, Turing reducibility. If the term "degree of unsolvability" is used without a further explanation, the *T*-degree is usually meant.

Definition 1. The *use function* $u(A;e,x,s)$ is 1+ (the maximum number used in computation if $\Phi_{e,s}^A(x)\downarrow$), and =0, otherwise. The use function $u(A;e,x)$ is $u(A;e,x,s)$ if $\Phi_{e,s}^A(x)\downarrow$ for some s, and is undefined if $\Phi_{e,s}^A(x)\uparrow$ (see Soare [1], p. 49).

Definition 2. Given a nonempty finite set $A = \{x_1,\ldots,x_k\}$, where $x_1 < x_2 < \ldots < x_k$, the number $y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_k}$ is the *canonical index of* A. Let D_y denote a finite set, with a canonical index y, and D_0 denote $\varnothing$ (see Soare [1], p. 33).

Definition 3. A is *computable in (Turing reducible to) B*, written $A \leq_T B$, if $A = \Phi_e^B$ for some *e* (Soare [1], p.50).

Definition 4. A is a *weak truth-table reducible to B*, written $A \leq_{wtt} B$, if $(\exists e)$ [$A = \Phi_e^B$ & (∃ computable f) [$D_{f(x)}$ contains all integers whose the membership or non-membership in *B* is used in the computation of $\Phi_e^B(x)$] (Rogers [2], p.158).

Definition 5. If *A* is a noncomputable computably enumerable (c.e.) set , then *a splitting of A* is a pair A_1, A_2 of disjoint c.e. sets such that $A_1 \cup A_2 = A$.

Definition 6. C.e. set A is *T-mitotic* (*wtt-mitotic*), if there is a splitting A_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$ ($A_1 \equiv_{wtt} A_2 \equiv_{wtt} A$) (Downey , Stob [3], p.83).

Definition 7. (*i*) A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a *strong array* if there is a recursive function f such that $F_n = D_{f(n)}$.

(*ii*) A strong array is *disjoint* if its members are pairwise disjoint.

(*iii*) An infinite set B is *hyperimmune*, abbreviated *h-immune*, if there is no disjoint strong array $\{F_n\}_{n\in\omega}$ such that $F_n \cap B \neq \varnothing$ for all n.

(iv) A c.e. set A is *hypersimple*, abbreviated *h-simple,* if $\overline{A}$ is *h*-immune (see Soare [1], p. 80).

Definition 8. A c.e. degree $\boldsymbol{a}$ is *contiguous* if for every pair A, B of c.e. sets in $\boldsymbol{a}$, $A \equiv_{wtt} B$ (Downey, Stob [3], p. 45).

Note that each contiguous degree, by definition, doesn't contain *T*-mitotic sets, which are not *wtt*-mitotic.

Lachlan proved the existence of non-mitotic c.e. set (Lachlan [4]).

Ladner proved the existence of completely mitotic c.e. degree (Ladner [5]).

Ladner and Sasso [6] proved that for every nonzero c.e. degree $\boldsymbol{b}$ there is a nonzero c.e. degree $\boldsymbol{a} \le \boldsymbol{b}$ such that $\boldsymbol{a}$ is contiguous (see also Downey, Stob [3]).

Thus, there is an infinite class of contiguous degrees, and these degrees, as we have mentioned, don't contain *T*-mitotic sets, which are not *wtt*-mitotic.

Ingrassia ([7]) proved the density of non-mitotic c.e. sets (in **R**) (see also Downey, Slaman, [8]).

II. Preliminaries, Basic Modules

Theorem 1. For any noncomputable c.e. degree $\boldsymbol{b}$ there is a degree $\boldsymbol{a}$ such that $\boldsymbol{a} \le \boldsymbol{b}$ and $\boldsymbol{a}$ contains a hypersimple *T*-mitotic, but not *wtt*-mitotic set.

Proof **(**sketch**).** Let h be a general computable function that maps ω to ω^2. Let (Ψ_i, ψ_i) denote the pair $(\Phi_{i_0}, \varphi_{i_0})$ for all i, where $h(i) = (i_0, i_1)$ (note that Ψ_i is *wtt*-reduction with ψ_i , denoting the corresponding use function).

It is known (Ladner [9]) that the computably enumerable set A is *T*-mitotic $\Leftrightarrow$ A is *T*-autoreducible, and similarly, the computably enumerable set A is *wtt*-mitotic $\Leftrightarrow$ A is *wtt*-autoreducible (Downey, Stob [3], see also Trakhtenbrot [10]).

Therefore, in order to achieve non-*wtt*-mitoticity, it is enough for us to achieve non-*wtt*-autoreducibility.

Thus, to prove our theorem, it is necessary to construct such a c.e. set A, so that the following requirements are met.

R_e: $(\exists x) \neg (\Psi_e(A \bigcup \{x\}; x) = A(x))$, if
$(\forall z \le x)\, (\psi_e(z)\downarrow \;\&\; u(A, e, z) \le \psi_e(z))$.

P_e: $[(\forall y)(\varphi_e(y)\downarrow) \;\&\; (\forall u, v)(u \ne v) \Rightarrow$
$D_{\varphi_e(u)} \bigcap D_{\varphi_e(v)} = \varnothing)] \Rightarrow (\exists z)(D_{\varphi_e(z)} \subseteq A)$.

Note that meeting the R_e requirement (for all e) provides the infinity of the set $\overline{A}$.

Order the requirements in the following priority ranking: $\mathrm{R}_0, P_0, \mathrm{R}_1, P_1, \ldots, \mathrm{R}_n, P_n, \ldots$.

Let $l(e,s) = \max\{x : (\forall y < x)(\Psi_{e,s}(A_s \bigcup \{y\} : y) = A(y)$ $\&\; (\forall z \le x)(\Psi_e(z)\downarrow))\}$.

The main strategy for satisfying R_e is as follows: we select a number (the so called follower) x (which should be accompanied by two more elements x - 2 and x - 1, and possibly, the third - $\hat{x}$; the exact definition of the attendant numbers of the follower x, namely $(x-2), (x-1), \hat{x}$, will be given hereinafter), we wait until $l(e,s) > x$ and enumerate x, in A_{s+1} , if $(\forall z < x)(\psi_{e,s}(z)\downarrow)$, setting $r(e, s+1) = u(x, e, s)$, where $u(x,e,s) = u(\Psi_{e,s}(A_s \bigcup \{x\}; x))$.

The *B*-permitting procedure is introduced in order to provide $A \le_T B$ (where B is a c.e. set from degree $\boldsymbol{b}$).

To meet the requirement of R_e at each stage, we have a finite set of followers $x_{1,s}, < \ldots < x_{n,s}$. In this construction, a modification of the B-permitting method is used. We treat the interval $[0, ..., i]$ as allowing for $x_{i,s}$.

To meet the requirement of P_e at each stage, we have a finite set of followers $y_{1,s}, < \ldots < y_{n,s}$. For requirement P_e, the usual B-permitting method is used.

The strategy for meeting a single requirement R_e in isolation (so called basic module for R_e), and also basic module for P_e will be given below.

A. Basic Module for R_e.

The followers $x_{1,s}, \ldots, x_{n,s}$ satisfy the following rules below.

Appointment. If $x_{i,s}$ is currently defined and $x_{i+1,s}$ is not, then if $l(e,s) > x_{i,s}$, declare $x_{i,s}$ as *active*, and set $x_{i+1,s+1} = \mu z(z \ge s+2 \;\&\; (\exists k)(z = 2k))$. Set $\tilde{r}(e, s+1) = max(u(x_{k,s}, e, s) : k \le i)$. To get an idea of the restriction function $\tilde{r}(e,s)$, we give its definition, although it is not used in the basic module.

We say that $x_{i,s}$ is *super-active*, if $x_{i,s} - 2$ and $x_{i,s} - 1$ belong to A_s .

Permission . If $x_{i,s}$ is active and $i \in B_{at\ s}$, then if

(i) $(\exists j > i)[x_{j,s}$ is super-active $\&\; x_{j,s} \notin A_s]$, let $j_0 = \mu z\, [x_{z,s}$ is super-active $\&\; x_{j,s} \in A_s]$. Then we enumerate the numbers $x_{j_0,s}$, $\hat{x}_{j_0,s}$ into A_{s+1} (where $\hat{x}_{j_0,s} = \psi_e(x_{j_0,s})$). Cancel $x_{k,s}$, for all $(k > j_0)$. We will do the same with the accompanying elements of the corresponding followers.

(ii) if (i) and $(\neg\exists j)[x_{j,s} \in A_s]$ does not hold, then we enumerate the numbers $x_{i,s} - 2$, $x_{i,s} - 1$ into A_{s+1}. Cancel $x_{k,s}$, for all ($k > i$). We will do the same with the accompanying elements of the corresponding followers.

For any i such that the follower $x_{i,s}$ is not canceled at the end of the part "permission" of the basic module and is active, let's set $x_{i,s+1} = x_{i,s}$. We will do the same with the accompanying elements of the corresponding followers.

The “cancellation” rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case it will be necessary to note the effect of the requirements of R_j and P_j (where $j < e$) on meeting the requirement R_e, but not to describe the basic module itself for R_e.

B. Basic Module for P_e.

The followers $y_{1,s},\dots,y_{n,s}$ satisfy the following rules below.

Appointment. If $y_{i,s}$ is currently defined and $y_{i+1,s}$ is not, then if $(\exists m)[\varphi_{e,s}(m)\downarrow \,\&\, (\forall z)[z \in D_{\varphi_{e,s}(m)} \Rightarrow z \ge y_{i,s}]]$, let m_0 be the least of such m.

Declare $y_{i,s}$ as *active* and $D_{\varphi_{e,s}(v)}$ as *a associated set of the active number* $y_{i,s}$. Also set $y_{i+1,s+1} = \mu z(z \ge s \,\&\, (\exists k)(z = 2k))$.

Permission. If $y_{i,s}$ is active and $i \in B_{at\ s}$, then enumerate the numbers $y_{i,s}, y_{i,s}+1$ and the set $D_{\varphi_{e,s}(m_0)}$ into A_{s+1} (where $D_{\varphi_{e,s}(m_0)}$ is an associated set of the active number $y_{i,s}$)

The “cancellation” rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case will be necessary to note the effect of the requirements of R_j (where $j \le e$) on meeting the requirement P_e, but not to describe the basic module itself for P_e.

The above rules are sufficient to safisfy a single R_e, and a single P_e, as we see below.

*Lemma 1.1***.** Suppose that ψ_e is total and $(\forall x)(\Psi_e(\mathrm{A}\bigcup\{x\};x)\downarrow \;\&\; u(A,e,x) \le \psi_e(x))$. Then $(\exists y)\neg((\Psi_e(A\bigcup\{y\};\, y) = A(y))$.

Thus, the requirement R_e is met.

Lemma 1.2. Suppose that φ_e is total. Then $[(\forall u,v)(u \ne v) \Rightarrow D_{\varphi_e(u)} \bigcap D_{\varphi_e(v)} = \varnothing)] \Rightarrow \; (\exists z)(D_{\varphi_e(z)} \subseteq A)$.

Thus, the requirement P_e is met.

These lemmas will be verified below.

III. Details

Verification of Lemma 1.1.

Suppose Lemma 1.1 is false.

We show that B is computable.

Note that since we only consider the satisfaction of the basis module for R_e (that is, we do not take into account the effect of the requirements R_j and P_j (where $j < e$) on the satisfaction of the requirement R_e), it is obvious that conditions (*i*), ... , (*iv*) are met.

(*i*) All the $x_{i,s}$ eventually become permanently defined, that is $\lim_s x_{i,s} = x_i$ exists with $x_i \notin A$.

(*ii*) Once x_k is defined at stage *t*, $(\forall s > t)(u(x_k,e,t) = u(x_k,e,s) = u(e,x_k))$.

(*iii*) $(\forall i)(x_{i+1} > \max\{u(e,x_k) : k \le i\})$.

(*iv*) It can be effectively recognized, when (*i*) occurs.

Two cases are possible:

(*a*) $(\exists\, m)\,(\forall\, k > m)\,[(x_k - 2) \notin A]$;

(*b*) $(\forall\, m)\,(\exists\, k > m)\,[(x_k - 2) \in A]$.

For both cases ((*a*) and (*b*)), it will be proved that B is computable (and thus, the assumption that Lemma 1 is false will lead to a contradiction with the supposition of noncomputability of B).

Now, if (a) holds, we prove that *B* is computable.

If conditions (*i*), ..., (*iv*) are satisfied, we show how to compute B (that is, the characteristic function of the set B; remind that we often identify the set B with its characteristic function).

Let $f \upharpoonright x$ denote the restriction of f to arguments $y < x$, and $A \upharpoonright x$ denotes $\chi_A \upharpoonright x$.

Let s_0 be such a stage that $B \upharpoonright m+1 = B_{s_0} \upharpoonright m+1$ and $A \upharpoonright x_{m+1} = A_{s_0} \upharpoonright x_{m+1}$.

Let $q \in \omega$ and $q > m$. Effectively compute a stage s so that x_{q+1} is defined, that is $x_{q+1} = x_{q+1,s}$ (in that case, in fact, $s > s_0$).

Then x_q is active, $x_q \in A$ and since x_{q+1} is the final value of the *q*+1-th follower, the computations of $u(x_j,e,s)$ are true for all $j \le q$.

In this case $q \in B \Leftrightarrow q \in B_s$, because otherwise it would lead to the fact that $x_q - 2$ would have entered the set *A*, contrary to our assumption that case (*a*) holds.

Now suppose that case (*b*) holds. Let us prove that in this case also B is computable.

If conditions (*i*), ..., (*iv*) are fulfilled, we show how to compute *B*.

Let $q \in \omega$. Effectively compute such a stage s and a number p so that $p = \mu z\ (z \ge q \;\&\; x_{z-2} \in A \;\&\; x_{z+1} = x_{z+1,s})$.

Then x_p is active, $x_p \notin A_s$ and since x_{p+1} is the final value of the $p+1$-th follower, then $u(x_j,e,s)$ computations are true for all $j \le p$. In this case $q \in B \Leftrightarrow q \in B_s$, since otherwise (that is, if q enters *B* after the stage *s*) this will lead to the entry p into A and satisfaction of the requirement R_e, which will contradict the initial assumption that Lemma 1 is false.

Lemma 1.1 is proved.

Verification of Lemma 1.2.

Suppose Lemma 1.2 is false.

Then all the $y_{i,s}$ become permanently defined (i.e. $\forall i\ \exists(t \ge t_0)\ (\forall s)\ (y_{i,t} = y_{i,s} = y_i))$ with $y_i \notin A$, since, otherwise, if $(\exists z,s)(y_{i,s} \in A)$, then the set $D_{\varphi_e(z)}$ (associated set of the active number $y_{i,s}$), according to the construction, would be included in A.

Assuming the opposite of the statement of the lemma, we show how B can be computed.

Let $q \in \omega$. Find $t \ge t_0$ such that y_q is permanently defined. Then $q \in B \Leftrightarrow q \in B_t$, since otherwise q's entry into B would meet P_e .

Lemma 1.2 is proved.

Note that the coherence of constructions to meet the requirements R_e and P_e (for all e) is not difficult, since meeting the requirements R_e and P_e (for all e) requires a finite number of steps.

We also note that the indicated method of constructing the set *A* (based on the constructions for the basic modules) will result in the set $A \cap \omega_{ev}$ being *T*-equivalent to the set $A \cap \omega_{od}$.

These remarks allow us to complete the proof of the theorem. ■

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