Efficiently Recognizable Sets, *P*-*T*-Mitoticity and Arithmetical Hierarchy

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***Abstract*— Let P be a class of problems recognized by a deterministic Turing machine which run in polynomial time. In parallel, the class $\widehat{\mathbf{P}}$ is considered. Indeed, the classes P and $\widehat{\mathbf{P}}$ are homomorphic (with respect to the relations in question). It is proved in the article that the index sets $\{ z \mid W_z$ is $\widehat{P}$ -*T*-mitotic$\}$, $\{z \mid W_z$ is weakly $\widehat{P}$ -*T*-mitotic$\}$, $\{ z \mid W_z$ is $\widehat{P}$ *T*-autoreducible$\}$ and $\{z \mid W_z \in \widehat{\mathbf{P}}\}$ are Σ_3-complete.**

***Keywords*—Arithmetical hierarchy, *P-T*-mitotic set, *P-T*-autoreducible set, index sets.**

I. Introduction

Information about the basic concepts of computability theory used in this article, in particular the Turing machine (TM), the numbering of computably enumerable sets $\{W_i\}_{i\in\omega}$ and the arithmetical hierarchy, can be found in Rogers [11], Soare [13].

Notation

Let ω be the set of all nonnegative integers
(i.e. $\omega = \{0,1,2,\cdots\}$).
Given a set Y, the set of all finite strings of elements from Y is denoted by Y*.
We fix the alphabet $\Lambda = \{0,1\}$.
The set Λ^* can be interpreted as binary representations of the natural number ω.

Cook [3] introduced the notion of polynomial time reducibility. This reducibility is just time bounded version of Turing reducibility ($\leq_T$) defined by Post [10].

A *Turing machine T* (*deterministic or nondeterministic*) *runs in polynomial time* if there is a polynomial function q such that for every input of length n any computation sequence of T halts in $q(n)$ or fewer moves.

A problem is simply a subset of Λ^* and **P** is the class of problems recognized by deterministic Turing machines, which run in polynomial time (see Ladner [9], p.155).

It is an intuitively appealing notion that **P** is the class of problems that can be solved efficiently.

In this article, we consider the class $\widehat{\mathbf{P}}$ (see below), such that the classes **P** and $\widehat{\mathbf{P}}$ are homomorphic (i.e., there is a homomorphic mapping from **P** into $\widehat{\mathbf{P}}$ and vice versa, there is a homomorphic mapping from $\widehat{\mathbf{P}}$ into **P**).

An *oracle Turing machine runs in polynomial time* if there exists a polynomial function q such that for *every* input of length n and *any* oracle set X, the machine halts within $q(n)$ steps (see Ladner [9], p.156).

Note that the definitions of R. Ladner [9] and other authors are based on the concept of a multitape Turing machine.

By analogy with the notions of *T*-mitoticity and *T*-autoreducibility, Ambos-Spies [1] introduced the notions of *P*-*T*-mitoticity, weakly *P*-*T*-mitoticity and *P*-*T*-autoreducibility.

This article studies the location of index sets $\{ z \mid W_z$ is $\widehat{\boldsymbol{P}}$ -*T*-mitotic$\}$, $\{z \mid W_z$ is weakly $\widehat{\boldsymbol{P}}$ -*T*-mitotic$\}$, $\{z \mid W_z$ is $\widehat{\boldsymbol{P}}$ -*T*-autoreducible$\}$ and $\{\boldsymbol{z} \mid W_z \in \widehat{\mathbf{P}}\}$ in the arithmetical hierarchy.

II. Preliminaries

Notation

We will denote the Λ^* elements by lower case Greek letters $\sigma, \tau, \dots$.
We let $\sigma\hat{}\tau$ denote the *concatenation* of string σ followed by τ.
Let $<$ be the natural order on Λ^* ($\lambda < 0 < 1 < 00 < 01 < \cdots$), where λ represents the empty string.
We will denote the subsets of Λ^* by upper case Greek letter $\Xi, \Theta, \dots$, as well as by the Latin letter P with subscripts (P_i).

If $\sigma \in \Lambda^*$, we let $|\sigma|$ denote the length of σ.
If $\Xi \subseteq \Lambda^*$, then

$$\Xi(\sigma) = \begin{cases} 1, & \text{if } \sigma \in \Xi \\ 0, & \text{if } \sigma \notin \Xi. \end{cases}$$

If $A \subseteq \omega$, then $A(x) = \chi_A(x)$, (where χ_A is a characteristic function of a set A.)

Define the mappings h_0, h_1 as follows:
Let h_0 be a 1-1 mapping from ω onto Λ^*,

$h_0(0) = \lambda$,
$h_0(n+1) = n+2$ -nd string according to the order of strings on Λ^*.

Let h_1 be a 1-1 mapping from Λ^* onto ω.
$h_1(\lambda) = 0$;
$h_1(n+1$-st string according to the order of strings on $\Lambda^*) = n$ (In fact, $h_1 = h_0^{-1}$).

REMARK. It can be proved that the mapping $h_1: \Lambda^* \to \omega$ is an *isomorphism (*see the definition of isomorphism in Waerden [17], pp. 25-26).

It is known that there exist effective enumerations of the sets $P_0, P_1, \ldots$ and oracle Turing machines $\boldsymbol{M}_0, \boldsymbol{M}_1, \ldots$, where P_i denotes the set recognized by the Turing machine (also denoted by P_i), which runs in polynomial time, and $\boldsymbol{M}_i$ denotes the oracle Turing machine, which runs in polynomial time. $\boldsymbol{M}_i(A)$ denotes the set recognized by $\boldsymbol{M}_i$ with the oracle A (see Ladner [9], p.157).

Notation.
$f \restriction x$ denotes the restriction of f to arguments y< x, and $A \restriction x$ denotes $\chi_A \restriction x$.
(Note that any string σ ∈ Λ* can be considered as a partial function from ω into Λ.)
Let $h_0(A) = \{\tau | (\exists x)\, [h_0(x) = \tau \ \&\ x \in A]\}$,

$h_1(\Xi) = \{x | (\exists \tau)[h_1(\tau) = x \ \&\ \tau \in \Xi]\}$.
Let $\{q_i\}_{i \in \omega}$ be the effective enumeration of polynomials.
Let $\hat{h}$ be a computable function from ω onto ω^2.

Based on the available numbering of computably enumerable (c.e.) sets $\{W_i\}_{i\in\omega}$, the available numbering of computable operators $\{\boldsymbol{\Phi}_i\}_{i\in\omega}$, and the available enumeration of polynomials we define for an arbitrary i (proceeding from the fact that $\hat{h}(i) = (i_0, i_1))$)

1) *the set* $\hat{P}_i$ *as follows*:
$$(\forall x)(\forall s \geq q_{i_1}(x)) \left[\hat{P}_{i,s}(x) = W_{i_0, q_{i_1}(x)}(x)\right],$$
2) *the oracle Turing machine* $\widehat{\boldsymbol{M}}_i$ *as follows*:
$$(\forall x)(\forall s \geq q_{i_1}(x)) \left(\forall \sigma_{|\sigma| \geq q_{i_1}}\right) [\widehat{\boldsymbol{M}}_{i,s}(\sigma)(x) = \boldsymbol{\Phi}_{i_0, q_{i_1}(x)}(\sigma \restriction_{q_{i_1}(x)})(x)].$$

Based on the known results (see, for example Hopcroft [6], Arora, Barak [2], Sipser [12], Terwijn [15]), the following conclusion is presented in Arora, Barak [2], p. 30:

All low-level choices (number of tapes, alphabet size, etc.) in the definition of Turing machines are immaterial, as they will not change the definition of **P**.

Thus, since neither the number of tapes nor the way the inputs and outputs are presented (binary coding or natural numbers) significantly affect (see, for example, Hopcroft [6], Arora, Barak [2]), *we can assert that*
$$(\forall i)(\exists j)(\forall x)\left[\hat{P}_i(x) = P_j(h_0(x))\right] \ \&$$
$$(\forall j)(\exists i)(\forall \sigma)\left[P_j(\sigma) = \hat{P}_i(h_1(\sigma))\right]$$
and
$$(\forall i)(\exists j)(\forall x)(\forall A)\left[\widehat{\boldsymbol{M}}_i(A)(x) = \boldsymbol{M}_j(h_0(A))(h_0(x))\right] \ \&$$
$$(\forall j)(\exists i)(\forall \sigma)(\forall \Xi)\left[\boldsymbol{M}_j(\Xi)(\sigma) = \widehat{\boldsymbol{M}}_i(h_1(\Xi))(h_1(\sigma))\right].$$

For a given numbering of c.e. sets $\{W_i\}_{i\in\omega}$ let
$\hat{P}Ind = \{z | (\exists i)[W_z = \hat{P}_i]\}$ and $\hat{\mathbf{P}} = \{\hat{P}_i\}_{i\in\omega}$.

Definition 1. Define $B \leq_T^P A$ if there is a such i that $B = \boldsymbol{M}_i(A)$ (see Ladner [9], Ambos-Spies [1]).

Definition 2. Define $B \leq_T^{\hat{P}} A$ if there is a such i that $B = \widehat{\boldsymbol{M}}_i(A)$.

Definition 3. A *splitting of* A is a pair A_1, A_2 of c.e. sets such that $A_1 \cap A_2$. We sometimes will write $A = A_1 \sqcup A_2$ if A_1, A_2 is a splitting of A (see Downey, Stob [5], p. 4).

Definition 4. A c.e. set A is *T-mitotic* if there is a splitting A_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$ (see Downey, Stob [5], p. 83, Lachlan [7], p. 9-10).

Let us recall some information about *T*-autoreducibility.

Definition 5. We say that a partial recursive functional $\boldsymbol{\Psi}$ is an *autoreduction* if, for all X and *n*, the computation of $\boldsymbol{\Psi}(X, n)$ includes no question of the form "$n \in X$?". A set A is *T-autoreducible* if there exists an autoreduction $\boldsymbol{\Psi}$ such that $A = \boldsymbol{\Psi}$(A) (see Trakhtenbrot [16], Ladner [8], p. 199).

From the definition of *T*-autoreducibility it follows that

A is T-autoreducible $\Leftrightarrow (\exists e)(\forall x)(\boldsymbol{\Phi}_e(A \cup \{x\})(x)) = A(x)) \Leftrightarrow (\exists e)(\forall x)(\boldsymbol{\Phi}_e(A - \{x\})(x)) = A(x))$.

Ambos-Spies introduced the following notions:

a) A computable set A is *P-T-mitotic* if there is a set $B \in \mathbf{P}$ such that $A \equiv_T^P A \cap B \equiv_T^P A \cap \bar{B}$. Otherwise, A is *non-P-T-mitotic* (see Ambos-Spies [1], p. 4).

b) A computable set A is *weakly P-T-mitotic* if there are sets A_0 and A_1 such that $A = A_0 \sqcup A_1$ and $A \equiv_T^P A_0 \equiv_T^P A_1$. Otherwise, A is *strongly non-P-T-mitotic* (see Ambos-Spies [1], p. 4).

c) A computable set A is *P-T-autoreducible* if for some $n \in \omega$ and every $x \in \Lambda^*$, $A(x) = \boldsymbol{M}_n(A - \{x\})(x)$ (see Ambos-Spies [1], p. 19).

(Ambos-Spice prefers the expression " $A(x) = \boldsymbol{M}_n(A - \{x\})(x)$ " instead of the equivalent expression "$A(x) = \boldsymbol{M}_n(A \cup \{x\})(x)$" in the definition of *P-T*-autoreducibility (see Downey, Slaman [4], p. 121).)

Ambos-Spies has proved that

(*i*) if A is *P-T*-mitotic, then A is *P-T*-autoreducible (see Ambos-Spies [1], p.19),

(*ii*) there is a computable set A, which is *P-T*-autoreducible, but not *P-T*-mitotic (see Ambos-Spies [1], p. 21).

We represent the definition of $\hat{P}$-*T*-mitoticity according to Ambos-Spies with slight modifications (see Ambos-Spies [1]).

Definition 6. A computable set A is *$\hat{P}$-T-autoreducible* if for some $n \in \omega$ and every, $x \in \omega$ $A(x) = \widehat{\boldsymbol{M}}_n(A \cup \{x\})(x)$.

Note that if A is cofinite or finite, then A is *P-T*-mitotic and, so A is weakly *P-T*-mitotic (see Ambos-Spies [1], pp. 4-5).

Definition 7. a) A computable set A is *$\hat{P}$-T-mitotic* if there is a set $B \in \widehat{\mathbf{P}}$ such that $A \equiv_T^{\hat{P}} A \cap B \equiv_T^{\hat{P}} A \cap \bar{B}$. Otherwise, A is *non-$\hat{P}$-T-mitotic*.

b) A computable set A is *weakly $\hat{P}$-T -mitotic* if there are sets A_0 and A_1 such that $A = A_0 \sqcup A_1$ and $A \equiv_T^{\hat{P}} A_0 \equiv_T^{\hat{P}} A_1$. Otherwise, A is *strongly non-$\hat{P}$-T -mitotic* (see Ambos-Spies [1], p. 4).

Definition 8. A relation $R \subseteq \omega^n$, $n \geq 1$, is *computable* if its characteristic function χ_R is computable, where $\chi_R(x_1, \cdots, x_n) = 1$ if $(x_1, \cdots, x_n) \in R$ and =0, otherwise (see Soare [13], p. 11).

Definition 9. (*i*) A set B *is in* Σ_0 , if B is computable,

(*ii*) For $n \geq 1, B$ *is in* Σ_n (written $B \in \Sigma_n$), if there is a computable relation $R(x, y_1, y_2, \cdots, y_n)$ such that $x \in B \Leftrightarrow (\exists y_1)(\forall y_2)(\exists y_3) \cdots (Q y_n) R(x, y_1, y_2, \cdots, y_n)$, where Q is $\exists$ if n is odd and $\forall$ if n is even (see Soare [13], p. 60).

Definition 10. For any given class $\mathcal{E}$ of computably enumerable sets, let $IND_{\mathcal{E}} = \{z | W_z \in \mathcal{E}\}$. If $A = IND_{\mathcal{E}}$ for some $\mathcal{E}$, A is called an *index set* (see Rogers [11], p. 324).

Definition 11. $Rec = \{z | W_z$ is computable (recursive)$\}$,
$Fin = \{z | W_z$ is finite$\}$,
$Cof = \{z | \overline{W}_z$ is finite$\}$ (see Soare [13], p. 17).

Definition 12. A set A is *Σ_n-complete* (*Π_n-complete*) if $A \in \Sigma_n$ (Π_n) and $B \leq_1 A$ for every $B \in \Sigma_n$ (Π_n) (it makes no difference whether we use "$B \leq_m A$" or "$B \leq_1 A$" in the definition of Σ_n-complete and Π_n-complete) (see Soare [13], p. 64).

It is known that Fin is Σ_2-complete, Cof and Rec are Σ_3 -complete (See Soare [13], pp. 65-67, Rogers [11], pp. 327-328).

Definition 13. $\hat{P}Ind = \{z \mid W_z \in \widehat{\mathbf{P}}\}$,
$T(\hat{P})M = \{z \mid W_z$ is $\hat{P}$-*T*-mitotic$\}$,
$WT(\hat{P})M = \{z \mid W_z$ is weakly $\hat{P}$-*T*-mitotic$\}$,
$AT(\hat{P}) = \{z \mid W_z$ is $\hat{P}$-*T*- autoreducible$\}$=
$\{z \mid (\exists i)(\forall x)[\widehat{\boldsymbol{M}}_i(W_z \cup \{x\})(x) = W_z(x)]$ & W_z is computable$\}$.

III. Results

In this paper, Proposition 1 and the following three Theorems are proved:

Proposition 1: The index set $\hat{P}Ind = \{z \mid W_z \in \widehat{\mathbf{P}}\}$ *is* Σ_3*-complete.*

Theorem 1: The index set $T(\hat{P})M = \{z \mid W_z$ *is* $\hat{P}$*-T- mitotic*$\}$ *is* Σ_3*-complete.*

Theorem 2: The index set $WT(\hat{P})M\{z \mid W_z$ *is weakly* $\hat{P}$*-T- mitotic*$\}$ *is* Σ_3*-complete.*

Theorem 3: The index set $AT(\hat{P})$=$\{z \mid W_z$ *is* $\hat{P}$-*T*-autoreducible$\}$ *is* Σ_3*-complete.*

IV. Conclusion

Studies of the locations of various index sets in the arithmetical hierarchy were carried out back in the 50s of the twentieth century (the works of H. Rice, N. Shapiro, H. Rogers and others are well known). In the following decades, these studies were actively continued thanks to the works of C. Yates, D. Martin, C. Jockusch, M.M. Arslanov, M. Stob, T. Slaman, R. Solovay, S. Schwarz and many others (see, for example, Soare [13], Chapter XII, Soare [14], Chapter 4).

In this article, the locations in the arithmetical hierarchy of the index sets, indicated in the *Abstract* of this article and in Chapter *Results* are precisely established.

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