On Singular Functional Variables in Hyperidentity of Distributivity

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***Abstract*—In this paper, we give a classification of non-trivial left and right distributive hyperidentities satisfied in non-trivial divisible algebras.**

***Keywords*— quasigroup, invertible algebra, division algebra, pre-cancellative algebra, hyperidentity, second-order formula**

I. Introduction

Hyperidentities [1]- [4] are second-order formulae [5] of the following form:

$$\forall X_1 ... \forall X_k \forall x_1 ... \forall x_n (w_1 = w_2),$$

where $X_1, ..., X_k$ are all function variables, and $x_1, ..., x_n$ are all object variables in the words (terms) w_1, w_2. The number k is called the functional rank of the hyperidentity. However, for simplicity, the hyperidentity (as well as the usual identity) is written without a quantifier prefix: $w_1 = w_2$. We will say that the hyperidentity $w_1 = w_2$ holds (is valid) in an algebra $(Q; \Sigma)$ if this equality is valid when every objective variable x_i is replaced by an arbitrary element of Q and every functional variable X_j is replaced by any operation of the corresponding arity from Σ (it is assumed that such a replacement is possible).

A hyperidentity

$$X(x, Y(y, z)) = U(V(x, y), Z(x, z))$$

is called a hyperidentity of left distributivity, but a hyperidentity

$$X(Y(z, y), x) = U(Z(z, x), V(y, x))$$

is called a hyperidentity of right distributivity. These hyperidentities are called dual.

A hyperidentity $w_1 = w_2$ is called functionally nontrivial if at least two different functional variables participate in it, i.e., when the functional rank of the hyperidentity is greater than 1. Hyperidentities of rank 1 are called trivial.

The binary algebra $(Q; \Sigma)$ is called:

1) distributive if it satisfies the following left and right distributive hyperidentities:

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)),$$

$$X(Y(x, y), z) = Y(X(x, z), X(y, z));$$

2) idempotent if it satisfies the idempotent hyperidentity:

$$X(x, x) = x;$$

3) commutative if it satisfies the commutativity hyperidentity:

$$X(x, y) = X(y, x);$$

4) hyperassociative if it satisfies the following associativity hyperidentity:

$$X(x, Y(y, z)) = Y(X(x, y), z)).$$

Let $a \in Q$, $X \in \Sigma$, define the following mappings: $L_{a,X}(x) = X(a, x)$, $R_{a,X}(x) = X(x, a)$ for all $x \in Q$. We say that $(Q; \Sigma)$ is a left (right) cancellative (divisible, invertible) algebra if the mappings $L_{a,X}$ ($R_{a,X}$) are injections (surjections, bijections) for all $X \in \Sigma$ and $a \in Q$. An algebra is called cancellative (divisible, invertible) if it is a left and right cancellative (divisible, invertible) algebra.

A groupoid $(Q; \cdot)$ is called left (right) pre-cancellative if

$$ca = cb \longrightarrow R_a = R_b,$$

$$(ac = bc \longrightarrow L_a = L_b),$$

where $a, b, c \in Q$. A groupoid is called pre-cancellative if it is both left and right pre-cancellative. A binary algebra $(Q; \Sigma)$ is called pre-cancellative if the groupoid $(Q; A)$ is pre-cancellative for any operation $A \in \Sigma$.

An algebra $(Q; \Sigma)$ is called left faithful if from $R_{a,X} = R_{b,X}$ follows $a = b$, where $a, b \in Q$, $X \in \Sigma$. A right faithful algebra is defined dually: from the equality $L_{a,X} = L_{b,X}$ follows $a = b$, where $a, b \in Q$, $X \in \Sigma$. An algebra is said to be faithful if it is both left and right faithful.

Obviously, an algebra is left (right) cancellative if and only if it is left (right) faithful and left (right) pre-cancellative.

II. Preliminary Results

Lemma 1: (i) The class of left (right) faithful algebras is closed under Cartesian products;

(ii) The class of left (right) pre-cancellative algebras is closed under Cartesian products.

Let $(Q; \Sigma)$ be a binary algebra. A non-empty subset $I \subseteq Q$ is called a right (left) ideal of the algebra $(Q; \Sigma)$ if for any operation $A \in \Sigma$ and for any $x \in Q$, $a \in I$ the following inclusions hold: $A(a, x) \in I$ ($A(x, a) \in I$). A two-sided ideal

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or just an ideal is a subset of $I \subseteq Q$ that is both a left and a right ideal.

Let $(Q;\Sigma)$ be a binary algebra. Let us introduce the following notation:

$$Id(Q) = \{x | \forall X \in \Sigma, X(x,x) = x\},$$

$$Id^{\exists}(Q) = \{x \in Q | \exists X \in \Sigma, X(x,x) = x\}.$$

Proposition 2: Let $(Q;\Sigma)$ be a distributive algebra. Then:

(1) If $Id(Q)$ is not empty, then $Id(Q)$ is an ideal of the algebra $(Q;\Sigma)$;

(2) $Id^{\exists}(Q) \neq \emptyset$ and is an ideal of $(Q;\Sigma)$;

(3) $X(x,Y(y,z)), X(Y(x,y),z) \in Id^{\exists}(Q)$ for all $X, Y \in \Sigma$, $x,y,z \in Q$.

Proof: (1) Let $x \in Id(Q)$ and $a \in Q$, then for any operations $X, Y \in \Sigma$ we have:

$$X(a,x) = X(a,Y(x,x)) = Y(X(a,x),X(a,x)),$$

thus, $X(a,x) \in Id(Q)$.

(2) For any $X \in \Sigma$ and for any $x \in Q$ we have $X(x,X(x,x)), X(X(x,x),x) \in Id^{\exists}Q$ because

$$X(x,X(x,x)) = X(X(x,x),X(x,x)) =$$

$$= X(X(x,X(x,x)),X(x,X(x,x)).$$

hence $Id^{\exists}Q$ is not empty. It is established similarly that $X(X(x,x),x) \in Id^{\exists}Q$.

For arbitrary elements $x \in Id^{\exists}Q$ and $a \in Q$ we show that $X(a,x), X(x,a) \in Id^{\exists}Q$ for all $X \in \Sigma$ operations. Since $x \in Id^{\exists}Q$, then there exists an operation $Y \in \Sigma$ such that $Y(x,x) = x$ and, therefore,

$$Y(X(a,x),X(a,x)) = X(a,Y(x,x)) = X(a,x).$$

It is shown in exactly the same way that $X(x,a) \in Id^{\exists}Q$. Thus, $Id^{\exists}Q$ is an ideal of the algebra $(Q;\Sigma)$.

(3) First, we show that $Y(x,X(x,x)), Y(X(x,x),x) \in Id^{\exists}Q$ for all $X, Y \in \Sigma$ and any $x \in Q$.

$$Y(x,X(x,x)) = X(Y(x,x),X(x,x)) =$$

$$= Y(X(x,Y(x,x)),X(x,Y(x,x))) =$$

$$= X(Y(X(x,Y(x,x)),x),Y(X(x,Y(x,x)),Y(x,x))) =$$

$$= X(X(Y(x,x),Y(Y(x,x),x)),Y(X(x,Y(x,x)),Y(x,x))),$$

given that $Y(Y(x,x),x) \in Id^{\exists}Q$ and $Id^{\exists}Q$ is an ideal, we obtain the inclusion $Y(x,X(x,x)) \in Id^{\exists}Q$. It can be proved similarly that $Y(X(x,x),x) \in Id^{\exists}Q$.

Let us now show that $X(x,Y(y,z)) \in Id^{\exists}(Q)$ for all $X, Y \in \Sigma$, $x,y,z \in Q$. Really,

$$Y(x,X(y,z)) = X(Y(x,y),Y(x,z)) =$$

$$= Y(X(Y(x,y),x),X(Y(x,y),z)) =$$

$$= Y(Y(X(x,x),X(y,x)),X(Y(x,y),z)) =$$

$$= Y(X(Y(X(x,x),y),Y(X(x,x),x)),X(Y(x,y),z)),$$

and given that $Y(X(x,x),x) \in Id^{\exists}Q$, we obtain the inclusion $Y(x,X(y,z)) \in Id^{\exists}Q$. It can be proved similarly that $Y(X(x,y),z) \in Id^{\exists}Q$ for any $X, Y \in \Sigma$ and $x,y,z \in Q$. $\square$

Since the divisible groupoid has no ideals, we obtain the following corollary.

Corollary 3: A distributive divisible groupoid is idempotent.

III. Main Result

A functional variable X is said to be singular in a hyperidentity $w_1 = w_2$ if X occurs just once in it and at least one of the following conditions holds:

1) a subword $w = X(\omega_1,\omega_2)$ involves object variables $x \in [\omega_1]$ and $y \in [\omega_2]$, each of which occurs just once in w;

2) a subword $w = X(\omega_1,\omega_2)$ has the form $X(\omega_1,x)$ or $X(x,\omega_2)$ and there exists an object variable $y \in [w]$ different from x and occuring just once in w, where $[w]$ is the set of object variables occurring in w.

A binary algebra $(Q;\Sigma)$ is called a d-algebra if Σ has a divisible operation, i.e., $Q(A)$ is a divisible groupoid for some $A \in \Sigma$.

Theorem 4: In a functionally non-trivial d-algebra with dual hyperidentities of left and right distributivity, a left or right distributivity hyperidentity does not have a singular functional variable.

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