A Hierarchy of Determinative Sequent Systems with Different Substitution Rules

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***Abstract*—Some determinative sequent system (DS) for classical propositional calculus is introduced on the base of well-known Tseytin's transformation. It is proved that the system DS is polynomial equivalent to the resolution system R and cut-free sequent system PK^-. Then we define the system SDS (DS with a substitution rule) and the systems S_kDS (DS with restricted substitution rules, where the number of connectives in substituted formulas is bounded by k). It is proved that for every $k \geq 0$ the system S_{k+1}DS has an exponential speed-up over the system S_kDS in the tree form, and the system SDS is polynomially equivalent to the Frege systems.**

Keywords*—*Determinative sequent systems, substitution rule, proof complexities, polynomial equivalence, exponential speed-up.

I. Introduction

The existence of a classical propositional proof system, which has polynomial size proofs for all tautologies, is equivalent to saying that NP = coNP [1]. This simple observation has drawn attention in recent years to the formalisms of propositional logic for the study of questions of computational complexity. A hierarchy of propositional proof systems has been defined by main complexity characteristics (size) and the relations between these systems are currently being analyzed. New systems are discovered and, as a consequence, the computational power of the old ones is better understood. In [2-5], it was shown that the addition of restricted cut rule, different substitution rules or quantifier rules to some propositional systems induces an exponential speed-up.

In this paper, some determinative sequent system (DS) for classical propositional calculus is introduced on the base of well-known Tseytin's transformation. It is proved that the system DS is polynomially equivalent to the resolution system R and cut-free sequent system PK^-. Then we define the system SDS (DS with a substitution rule) and systems S_kDS (DS with restricted substitution rules, where the number of connectives in substituted formulas is bounded by k). It is proved that for every $k \geq 0$, the system S_{k+1}DS has an exponential speed-up over the system S_kDS in the tree form, and the system SDS is polynomially equivalent to the Frege systems.

II. Preliminaries

To prove our main result, we recall some notions and notations from [1-4]. We will use the current concepts of the unit Boolean cube (E^n), a propositional formula, a tautology, a proof system for propositional logic and proof complexity. The language of the considered systems contains propositional variables, logical connectives $\neg, \&, \vee, \supset$ and parentheses (,).

Considered Proof Systems

a) Resolution system R proves a formula to be a tautology by showing that its negation, which is put into conjunctive normal form, is unsatisfiable. A *literal* is a propositional variable p or a conjugate $\bar{p}$. A *clause* is a finite set of literals, where the meaning of the clause is the disjunction of the literals in the clause. Resolution has no fixed axioms. It has only one **resolution rule**:

$$\frac{C_1 \cup \{p\} \quad C_2 \cup \{\bar{p}\}}{C_1 \cup C_2}.$$

For every formula, by the well-known Tseytin's transformation, we can obtain some unsatisfiable set of clauses $\mathfrak{C}$, which is considered as a set of axioms, to which we apply the resolution rule until the empty clause is obtained.

Substitution rule for the set of clauses $\mathfrak{C}$ is introduced as follows: $\frac{\mathfrak{C}}{S(\mathfrak{C})_p^A}$, where $S(\mathfrak{C})_p^A$ denotes the set of results of substitution of the formula A instead of the variable p everywhere in the clauses of the set $\mathfrak{C}$, and ***generalized* resolution rule** for the formula A

$$\frac{C_1 \cup \{A\} \quad C_2 \cup \{\bar{A}\}}{C_1 \cup C_2},$$

where A is a literal or a substituted formula. By **SR** we denote the system R with the substitution rule and generalized resolution rule. If the number of connectives of substituted formulas is bounded by k, then the corresponding system is denoted by **S_kR**.

b) The system E (elimination system) was described in [1], where some new notions were introduced.

Let φ be a propositional formula, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of φ, and $P' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\}$ $(1 \leq m \leq n)$ be some subset of P.

The conjunct K can be represented simply as a set of literals (no conjunct contains a variable and its negation simultaneously).

Definition 2.1.1. Given $=\{\sigma_1,\sigma_2,\dots,\sigma_m\}\subset E^m$, the conjunct $K^\sigma=\{p_{i_1}^{\sigma_1},p_{i_2}^{\sigma_2},\dots,p_{i_m}^{\sigma_m}\}$ is called φ − 1-determinative (φ − 0-determinative) if assigning σ_j $(1\le j\le m)$ to each p_{i_j} we obtain the value of φ $(1\ or\ 0)$ independently of the values of the remaining variables.

Definition 2.1.2. DNF $D=\{K_1,K_2,\dots,K_r\}$ is called a determinative disjunctive normal form (dDNF) for φ if $\varphi=D$ and every conjunct K_i $(1\le i\le r)$ is 1-determinative for φ.

The axioms of E are not fixed, but for every formula φ each conjunct from some dDNF of φ can be considered as an axiom.

The elimination rule (ε-rule) infers $K'\cup K''$ from the conjunct $K'\cup\{p\}$ and $K''\cup\{\bar{p}\}$, where K' and K'' are conjuncts and p is a variable.

DNF $D=\{K_1,K_2,\dots,K_l\}$ is called full (tautology) if, using the ε-rule, one can prove the empty conjunction ($\emptyset$) from the axioms $\{K_1,K_2,\dots,K_l\}$.

By the analogy to the systems SR and S_kR, we can introduce the systems **SE** with a substitution rule and a generalization of the ε-rule, and **S_kE** with restricted substitution rules.

c) By PK is denoted the usual sequent system LK, where the rules are restricted to propositional logic. By **PK⁻** is denoted the sequent system PK without a cut rule. By **PK^k** is denoted the sequent system PK with a cut rule, where the number of connectives of cut formulas is bounded by k [6].

d) The Frege system $\mathcal{F}$ uses a denumerable set of propositional variables, a finite, complete set of propositional connectives; $\mathcal{F}$ has a finite set of inference rules defined by a figure of the form $\frac{A_1A_2\dots A_m}{B}$ (the rules of inference with zero hypotheses are the axioms schemes); $\mathcal{F}$ must be sound and complete, i.e., for each rule of inference $\frac{A_1A_2\dots A_m}{B}$ every truth-value assignment, satisfying $A_1A_2\dots A_m$, also satisfies B, and $\mathcal{F}$ must prove every tautology.

Proof Complexity

By $|\varphi|$ we denote the size of the formula φ, defined as the number of all logical sign entries in it. It is obvious that the full size of the formula, which is understood to be the number of all symbols is bounded by some linear function in $|\varphi|$.

The proof complexities are considered for comparison of different proof systems. In the theory of proof complexity, the main characteristic of the proof is l– complexity, which is the *size* of a proof (= the sum of all formulae sizes). The minimal l–complexity of a formula φ in a proof system Φ we denote by $l^\Phi(\varphi)$.

Definition 2.2.1. The system Φ_1 *p-simulates* the system Φ_2 if there exists a polynomial $p()$ such that for each formula φ provable both in the systems Φ_1 and Φ_2, we have $l^{\Phi_1}(\varphi)\le p\big(l^{\Phi_2}(\varphi)\big)$.

Definition 2.2.2. The systems Φ_1 and Φ_2 are *p-equivalent*, if systems Φ_1 and Φ_2 *p-simulate* each other.

Definition 2.2.3. If Φ_2 *p-simulates* the system Φ_1 and for some sequence of formulas φ_n in the two systems Φ_1 and Φ_2 for sufficiently large n is valid $l^{\phi_1}(\varphi_n)=\Omega(2^{l^{\phi_2}(\varphi_n)})$), then we say that the system ϕ_2 has an exponential speed-up over the system ϕ_1.

III. Main results

We construct some **determinative sequent system DS** for classical propositional calculus using Tseytin's transformation in sequent form and investigate their relationships with other proof systems.

Let φ be some formula and $\{p_1,p_2,\dots,p_n\}$ be the set of its distinct variables (later we call them main variables). We will associate a new variable with every non-elementary subformula of φ, where the negation of the subformula will be associated with the negation of the corresponding variable. We construct the system of determinative sequents as follows.

1. If α,β,γ are variables associated correspondingly with the subformulas $B\vee C$, B, C, then the system of sequents will be $\{\beta\to\alpha;\ \gamma\to\alpha;\ \bar{\beta},\bar{\gamma}\to\bar{\alpha}\}$.
2. For the subformula $B\&C$, the system of sequents will be $\{\bar{\beta}\to\bar{\alpha};\ \bar{\gamma}\to\bar{\alpha};\ \beta,\gamma\to\alpha\}$.
3. For the subformula $B\supset C$, the system of sequents will be $\{\bar{\beta}\to\alpha;\ \gamma\to\alpha;\ \beta,\bar{\gamma}\to\bar{\alpha}\}$.

The axioms of DS are not fixed, but for every formula φ each above described determinative sequent can be considered as an axiom.

The rules of inference are:

$$\text{cut rule: } \frac{\Gamma\to p\quad p\to\Delta}{\Gamma\to\Delta};$$

$$\frac{\Gamma\to\Delta,p}{\bar{p},\Gamma\to\Delta};\qquad\qquad \frac{p,\Gamma\to\Delta}{\Gamma\to\Delta,\bar{p}},$$

where p is a literal and if p is a negation of some variable, then $\bar{p}$ is a variable itself.

If the formula φ is associated with a variable s, then the sequent $\to s$ will prove in DS iff the formula φ is a tautology.

Theorem 1. The system DS is p-equivalent to the system E.

Proof sketch. Let P be a tree-like proof of $\to s$ corresponding to the formula φ in the system DS with the minimal size. We can construct φ-determinative conjuncts as follows. For every path i between two vertices, one of which is associated with an axiom and the other with a sequent $\to s$, we construct a conjunct K_i as the set of all main variables occurring in the sequents of this path. On top of all these φ-determinative conjuncts, we can construct a φ-proof in the system E. In the opposite direction, the proof can be given by analogy with the proof, given in the second paragraph of [2].

Theorem 2. The system DS is p-equivalent to the resolution system R.

The proof is obvious, taking into consideration that i) axioms in both systems created by Tseytin's transformation are like the systems of disjuncts in R and the systems of sequents in DS, ii) the inference rules are "equivalent" to each other.

If we add a substitution rule to the system DS, we can also introduce the corresponding *generalized* rules for a formula A:

$$\frac{\Gamma \to A \quad A \to \Delta}{\Gamma \to \Delta};$$

$$\frac{\Gamma \to \Delta, A}{\bar{A}, \Gamma \to \Delta}; \qquad \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \bar{A}},$$

where A is a literal or a substituted formula. By SDS we denote the system DS a with substitution rule and the above generalized rules. If the number of connectives of substituted formulas (therefore cut formula) is bounded by k then the corresponding system is denoted by S_kDS.

From the above theorems and results of [4], we can prove the following corollaries:

Proposition 1. The system DS is p-equivalent to the system PK^-.

Proposition 2. For every $k \geq 0$, the system S_kDS is p-equivalent to the system PK^k, and $S_{k+1}DS$ has an exponential speed-up over the system S_kDS in the tree form.

Proposition 3. The system SDS is p-equivalent to the system PK.

Proposition 4. The system SDS is p-equivalent to the Frege systems.

IV. Conclusion

Tseytin's transformation allows for every propositional formula φ to construct a system of axioms (disjuncts, sequents), on the base of which one can check whether the formula is a tautology or not. The sum of all such axiom sizes is no more than $6|\varphi|$. If we consider the disjuncts from conjunctive normal form for the negation of φ or determinative conjuncts from dDNF of φ as axioms, then the sum of axiom sizes can be exponentially larger than $|\varphi|$, but our results show that the size of proofs in both cases are nearly the same.

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