A New Sufficient Condition for a 2-Strong Digraph to Be Hamiltonian

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***Abstract*—In this paper we prove:**
Let D be a 2-strong digraph of order $n \geq 9$. If $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex has degree at least $n-k-4$, where k is a non-negative integer, then D is Hamiltonian.

This result is an extension of Ghouila-Houri's theorem for 2-strong digraphs and is in some sense the best possible.

We also give a new sufficient condition for a digraph to be Hamiltonian-connected.

***Keywords*— Digraph, Hamiltonian cycle, k-strong, degree.**

I. Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [1]. Every cycle and path is assumed simple and directed. A cycle (a path) in a digraph D is called *Hamiltonian* (*Hamiltonian path*) if it includes all the vertices of D. A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle. Hamiltonicity is one of the most central in graph theory and has been extensively studied by numerous researchers. The problem of determining the Hamiltonicity of a graph (digraph) is NP-complete, but there are numerous sufficient conditions which ensure the existence of a Hamiltonian cycle in a digraph (see [1]–[4]). Among them are the following classical sufficient conditions for a digraph to be Hamiltonian.

Theorem 1: (Nash-Williams [5]). Let D be a digraph of order $n \geq 2$. If for every vertex x of D, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.

Theorem 2: (Ghouila-Houri [6]). Let D be a strong digraph of order $n \geq 2$. If for every vertex x of D, $d(x) \geq n$, then D is Hamiltonian.

Theorem 3: (Woodall [7]). Let D be a digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of distinct vertices x and y of D such that there is no arc from x to y, then D is Hamiltonian.

Theorem 4: (Meyniel [8]). Let D be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent distinct vertices x and y of D, then D is Hamiltonian.

It is known that all the lower bounds in the above theorems are tight. Notice that for strong digraphs, Meyniel's theorem is a generalization of Nash-Williams', Ghouila-Houri's and Woodall's theorems.

Nash-Williams [5] suggested the problem of characterizing all the strong digraphs of order n and minimum degree $n-1$ that have no Hamiltonian cycle. As a partial solution of this problem, Thomassen proved a structural theorem on the extremal digraphs, in an excellent paper [9]. An analogous problem for the Meyniel theorem was considered by the author [10], proving a structural theorem on strong non-Hamiltonian digraphs D of order n with the condition that $d(x) + d(y) \geq 2n-2$ for every pair of non-adjacent distinct vertices x, y. This improves the corresponding structural theorem of Thomassen. In [10], it was also proved that if m is the length of longest cycle in D, then D contains cycles of all lengths $k = 2, 3, \ldots, m$. Thomassen [9] and the author [11] described all the extremal digraphs for the Nash-Williams theorem, respectively, when the order of the digraph D is odd and when the order of the digraph D is even. Here we combine them in the following theorem.

Theorem 5: (Thomassen [9] and Darbinyan [11]). Let D be a digraph of order $n \geq 4$ with minimum degree $n-1$. If for every vertex x of D, $d^+(x) \geq n/2-1$ and $d^+(x) \geq n/2-1$, then D is Hamiltonian, except some exceptions, which are completely characterized.

Goldberg et al. [12] relaxed the condition of the Ghouila-Houri theorem by proving the following theorem.

Theorem 6: (Goldberg et al. [12]). Let D be a strong digraph of order $n \geq 2$. If $n-1$ vertices of D have degrees at least n and the remaining vertex has degree at least $n-1$, then D is Hamiltonian.

Note that Theorem 6 is an immediate consequence of Theorem 4. In [12], the authors for any $n \geq 5$ presented two examples of non-Hamiltonian strong digraphs of order n such that: (i) In the first example, $n-2$ vertices have degrees equal to $n+1$ and the other two vertices have degrees equal to $n-1$. (ii) In the second example, $n-1$ vertices have degrees at least n and the remaining vertex has degree equal to $n-2$.

It is worth mentioning that, Thomassen [9] constructed a strong non-Hamiltonian digraph of order n with only two vertices of degree $n-1$ and all other $n-2$ vertices have degrees at least $(3n-5)/2$.

In [13], Zhang et al. reduced the lower bound in Theorem 3 by 1, and proved that the conclusion still holds with only a few exceptional cases that can be clearly characterized. In [14], we showed that: Meyniel's theorem remains true if we reduce the lower bound in Theorem 4 by one, for only one pair of non-adjacent distinct vertices.

In [15], it was announced that the following holds:

Theorem 7: Let D be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least n and the remaining vertex has degree at least $n-4$. Then D is Hamiltonian.

The proof of Theorem 7 has never been published. In [16], we presented the proof of the first part of Theorem 7, by proving the following:

Theorem 8: Let D be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least n and the remaining vertex z has degree at least $n-4$. If D contains a cycle of length $n-2$ passing through z, then D is Hamiltonian.

In [16], we also proposed the following conjecture.

Conjecture 1. *Let D be a 2-strong digraph of order n. Suppose that $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex has degree at least $n-k-4$, where k is a non-negative integer. Then D is Hamiltonian.*

Note that for $k=0$, this conjecture is Theorem 7. Recently, we have settled Conjecture 1 for any non-negative integer k by proving the following theorem.

Theorem 9: Let D be a 2-strong digraph of order $n \geq 9$ such that $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex z has degree at least $n-k-4$, where k is a non-negative integer. Then D is Hamiltonian.

In [17], we gave the proof of the first part of Theorem 9 for any positive integer k, which we formulate as Theorem 10.

Theorem 10: Let D be a 2-strong digraph of order $n \geq 3$ such that $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex z has degree at least $n-k-4$, where k is a positive integer. If the length of a longest cycle through z is at least $n-k-2$, then D is Hamiltonian.

The goal of this work is to present the complete proof of the second part of the proof of Theorem 9.

In addition, we also show that Theorem 9 is some sense the best possible. Using Theorem 9, we can prove that the following theorem holds, which is an analogue of the Overbeck-Larisch theorem [18].

Theorem 11: Let D be a 3-strong digraph of order $n \geq 10$ with the minimum degree at least $n+k+2$, where $k \geq 0$ is an integer. If for two distinct vertices u and v, the following holds: $d^+(u)+d^-(v) \geq n-k-2$ or $uv \notin A(D)$ and $d^+(u)+d^-(v) \geq n-k-4$, then D has a Hamiltonian (u,v)-path.

II. Sketch of the Proof of Theorem 9

By contradiction, suppose that D is not Hamiltonian. Then we know that D has no $C(z)$-cycle of length greater than $n-k-3$ through z and D contains a cycle $C_{n-1} := x_1x_2 \ldots x_{n-1}x_1$ of length $n-1$ such that $z \notin V(C_{n-1})$. There are two distinct vertices, say x_1 and x_{n-d-1} such that $x_{n-d-1} \to z \to x_1$ and z is not adjacent to any vertex x_i with $n-d \leq i \leq n-1$. For any $i \in [1,d]$, let $y_i = x_{n-d-1+i}$, $Y = \{y_1, y_2, \ldots, y_d\}$ and let P denote the path $x_1x_2 \ldots x_{n-d-1}$. We first show that Claims 1-4 are true.

Claim 1. *Suppose that $D\langle Y \rangle$ is strong and each vertex y_j of Y cannot be inserted into P. If $d^+(x_i, Y) \geq 1$ with $i \in [1, n-d-2]$, then $A(Y \to \{x_{i+1}, x_{i+2}, \ldots, x_{n-d-1}\}) = \emptyset$.*

Claim 2. *If $x_j \to z$ with $j \in [1, n-d-2]$, then $A(z \to \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = \emptyset$.*

Claim 3. *Suppose that there is $l \in [2, n-d-2]$ such that $A(\{x_1, x_2, \ldots, x_{l-1}\} \to Y) = A(Y \to \{x_{l+1}, x_{l+2}, \ldots, x_{n-d-1}\}) = \emptyset$. Then for every $j \in [2, n-d-2]$,*

$$A(\{x_1, x_2, \ldots, x_{j-1}\} \to \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) \neq \emptyset.$$

Claim 4. *Any vertex y_j cannot be inserted into P.*

To complete the proof of Theorem 9, we distinguish two cases according to the subdigraph $D\langle Y \rangle$ is strong or not. Note that when $D\langle Y \rangle$ is not strong, then $k=0$ and $d=4$. When $D\langle Y \rangle$ is strong, then we need to prove some additional claims and consider several subcases. □

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