# On One Connection Between the Moments of Random Variables

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***Abstract*—A new elementary proof of one result of R. Fukuda is proposed. Some improvements are also presented.**

***Keywords*— probability space, random variable, mathematical expectation.**

## I. Introduction

Let us start with formulation of an interesting result of S. Banach proved in 1933. This result is not used in this paper directly but it will help us to better discuss the problem.

*Theorem 1:* (S. Banach, [1]) From any bounded orthonormal system $(\varphi_n)$ a subsystem $(\varphi_{n_k})$ can be chosen so that the series $\sum_{k=1}^{\infty} \alpha_k \varphi_{n_k}$ converge in $L_p([0,1])$ for any $p$, $1 \le p < \infty$, whenever $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$.

In other words, from any bounded orthonormal system a subsystem can be chosen, which is $p$-lacunary for any $p$, $1 \le p < \infty$.

Another formulation of this result is as follows: there exists an isomorphism of the Hilbert space $l_2$ onto the closed linear manifold $L$ in $L_p([0,1])$ spanned by the functions $\varphi_{n_k}$, $k = 1, 2, \ldots$.

Under this isomorphism the unit vectors $e_k = (0, \ldots, \overset{k}{1}, 0, \ldots)$, in $l_2$ correspond to the functions $\varphi_{n_k}$, i.e. the basic sequence $\varphi_{n_k}$ is equivalent to the natural basis $(e_k)$ in $l_2$. The existence of this isomorphism implies that there exists a constant $C \ge 1$ (the norm of the isomorphism), depending only on $L$ and $p, 1 \le p < \infty$, such that for any $x \in L$ we have

$$\left( \int_0^1 |x(t)|^p \, dt \right)^{1/p} \le C \left( \int_0^1 |x(t)|^2 \, dt \right)^{1/2}.$$

Note that for $1 \le p \le 2$ the statement of the theorem is obvious (and $C = 1$ for this case).

Inspired by this result of S. Banach, in 1962 M.I. Kadec and A. Pelczynski [2] investigated more general version of the Banach theorem. In particular, for any sequence $(x_n)$ in $L_p(0,1), p > 2$, they found the necessary and sufficient condition on $(x_n)$ to contain a basic sequence $(x_{n_k})$ equivalent to the natural basis $(e_k)$ in $l_2$.

## II. Main Result

Investigating the Subgaussian random elements with values in Banach spaces and analyzing the results of [2], R. Fukuda [3] came to a result, which is improved in our Theorem 2 stated below.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\xi : \Omega \to \mathbb{R}^1$ be a real random variable and $\mathbb{E}$ be a mathematical expectation symbol.

*Theorem 2:* Let $p > q > 0$ and for some $C \ge 1$

$$\{\mathbb{E}|\xi|^p\}^{1/p} \le C \{\mathbb{E}|\xi|^q\}^{1/q} < \infty. \tag{1}$$

Then for any $r, s$, $0 < r, s \le p$, we have

$$\{\mathbb{E}|\xi|^r\}^{1/r} \le C^\beta \{\mathbb{E}|\xi|^s\}^{1/s},$$

where

$$\beta = \begin{cases} 0, & \textit{if} \quad 0 < r \le s \le p, \\ 1, & \textit{if} \quad q \le s < r \le p, \\ \frac{q(p-s)}{s(p-q)}, & \textit{if} \quad 0 < s < q < r \le p, \\ \frac{p(q-s)}{s(p-q)}, & \textit{if} \quad 0 < s < r \le q. \end{cases}$$

*Proof:* Since the expression $\{\mathbb{E}|\xi|^t\}^{1/t}$, as a function of $t$, $t > 0$, is nondecreasing, the statement of the theorem for the case $0 < r \le s \le p$ is evident. For the case $q \le s < r \le p$, the proof is also easy using the condition (1) of the theorem in addition. Therefore, we begin the proof with the case $0 < s < q < r \le p$. Introduce the numbers $u = \frac{p(q-s)}{q(p-s)}$ and $v = \frac{s(p-q)}{q(p-s)}$. Clearly $0 < u, v < 1$ and $u + v = 1$. Using the Hölder inequality, we get the following inequality:

$$\mathbb{E}|\xi|^q = \mathbb{E}|\xi|^{uq}|\xi|^{vq} \le \left\{\mathbb{E}|\xi|^{uqt}\right\}^{1/t} \left\{\mathbb{E}|\xi|^{vqt^\star}\right\}^{1/t^\star}, \tag{2}$$

where $1 < t, t^\star < \infty$ and $1/t + 1/t^\star = 1$. Choose now $t$ by the condition $uqt = p$. It is clear that $t > 1$ and

$$t = \frac{p-s}{q-s}, \qquad t^\star = \frac{t}{t-1} = \frac{p-s}{p-q}.$$

For such a number $t$, the relation (2) leads to the following one

$$\mathbb{E}|\xi|^q \le \{\mathbb{E}|\xi|^p\}^{\frac{q-s}{p-s}} \{\mathbb{E}|\xi|^s\}^{\frac{p-q}{p-s}} \le$$

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$$\leq C^{\frac{p(q-s)}{p-s}} \left\{\mathbb{E}|\xi|^q\right\}^{\frac{p(q-s)}{q(p-s)}} \left\{\mathbb{E}|\xi|^s\right\}^{\frac{p-q}{p-s}},$$

from which the following inequality, the key point for our proof, can easily be obtained

$$\left\{\mathbb{E}|\xi|^q\right\}^{1/q} \leq C^{\frac{p(q-s)}{s(p-q)}} \left\{\mathbb{E}|\xi|^s\right\}^{1/s}. \qquad (3)$$

Using now the Hölder inequality, the assumption of the theorem and finally, the key inequality (3), we get:

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq \left\{\mathbb{E}|\xi|^p\right\}^{1/p} \leq C \left\{\mathbb{E}|\xi|^q\right\}^{1/q} \leq$$
$$\leq C^{\frac{q(p-s)}{s(p-q)}} \left\{\mathbb{E}|\xi|^s\right\}^{1/s}.$$

Now the case $0 < s < r \leq q$ is left, which can be reduced to the previous one. □

Note that applying Kahane's inequality, Fukuda in his paper [3] as a constant $C^\beta$ for $r = p$ and $s = 1$ obtained the expression

$$C^{1+\frac{pq}{p-q}} \cdot q \cdot B^{-1}(1/q, \frac{p}{p-q} + 1), \qquad (4)$$

where $B(\cdot,\cdot)$ is a beta function. As our calculations show, in some partial cases the constant $C^\beta$ obtained by us is better than (4) and, moreover, it reflects its dependence on $p, q, r, s$ and $C$ more clearly.


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