Complete Caps in $AG(n,\ 3)$

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***Abstract*—A cap in a projective or affine geometry over a finite field is a set of points no three of which are collinear. We give some new constructions for complete caps in affine geometry $AG(n,\ 3)$, some of which have maximal possible sizes.**

***Keywords*— Affine geometry, points, caps, complete caps.**

I. Introduction

The main problem in the theory of caps is to find the minimal and maximal sizes of complete caps in projective geometry $PG(n,\ q)$ or in affine geometry $AG(n,\ q)$. Determining the exact value of the minimum and maximum cardinality of caps in projective geometry $PG(n,\ q)$ or in affine geometry $AG(n,\ q)$ seems to be a very hard problem. There are some well-known constructions (doubling, product and recursive), which allow to create large high-dimensional caps based on large low-dimensional caps. Note that the problem of determining the complete cap of the minimum size is of particular interest in the Coding theory. In this paper, we consider the problem of constructing complete caps in affine geometry $AG(n,\ 3)$ over the field $F_3 = \{0,\ 1,\ 2\}$. A cap is a set of points, no three of which are collinear. A cap is called complete when it cannot be extended to a larger one. Let us denote the size of the largest caps in $AG(n,\ q)$ and $PG(n,\ q)$ by $c_{n,q}$ and by $c'_{n,q}$, respectively. Presently, only the following exact values are known: $c_{n,2} = c'_{n,2} = 2^n,\ c_{2,q} = c'_{2,q} = q+1$ if q is odd, $c_{2,q} = c'_{2,q} = q+2$ if q is even, and $c_{3,q} = q^2,\ c'_{3,q} = q^2+1$ [1,2]. Apart from these general results, the exact values are known in the following cases: $c_{4,3} = c'_{4,3} = 20$ [3], $c'_{5,3} = 56$ [4], $c_{5,3} = 45$ [5], $c_{4,4} = 40,\ c'_{4,4} = 41$ [6], $c_{6,3} = 112$ [7]. In the other cases, only lower and upper bounds on the sizes of caps in $AG(n,\ q)$ and $PG(n,\ q)$ are known [8, 9, 10]. In this paper, we give some new constructions for complete caps in affine geometry $AG(n,\ 3)$ implying some well-known results.

II. Main Results

It is not difficult to see that if C_n is a cap in $AG(n,\ 3)$, then $\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} \neq \mathbf{0}(mod\ 3)$ for every triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in C_n$. For the point $\boldsymbol{\alpha} \in AG(n,\ 3)$, let's denote

$$\alpha(0) = \{i\ |\alpha_i = 0,\ i \in [1,\ n]\},$$
$$\alpha(1) = \{i\ |\alpha_i = 1,\ i \in [1,\ n]\},$$
$$\alpha(2) = \{i\ |\alpha_i = 2,\ i \in [1,\ n]\},$$
$$X(\boldsymbol{\alpha}) = \{\boldsymbol{x}\ |\boldsymbol{x} \in AG(n,\ 3),\ \boldsymbol{x}(0) = \boldsymbol{\alpha}(0)\}.$$

Further, let

$$B_n = \{\boldsymbol{\alpha} = (\alpha_1, ...,\ \alpha_n) \mid \alpha_i = 1, 2\},$$
$$B'_n = \{\boldsymbol{\alpha} \in B_n \mid |\boldsymbol{\alpha}(1)|\ is\ odd\},$$
$$B''_n = \{\boldsymbol{\alpha} \in B_n \mid |\boldsymbol{\alpha}(1)|\ is\ even\}.$$

In 2015, the second author [11] introduced the concept of P_n-set, which we use in our research. Notice that the set of the points $A \subseteq AG(n,\ 3)$ is called a P_n-set, if it satisfies the following two conditions:
$i)$ for any two distinct points $\boldsymbol{\alpha},\boldsymbol{\beta} \in A$, there exists such i that $\alpha_i = \beta_i = 0$, where $1 \leqslant i \leqslant n$,
$ii)$ for any triple of distinct points $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma} \in A,\ \boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma} \neq \mathbf{0}(mod\ 3)$.

Throughout this article, the notation $P_n(P_{n_i})$ will mean P_n-set (P_{n_i}-set). We call P_n to be complete when it cannot be extended to a larger one. By the mirror inversion of the point $\boldsymbol{\alpha} = (\alpha_1, ...,\ \alpha_n)$ we mean the point $\hat{\boldsymbol{\alpha}} = (\alpha_n, \alpha_{n-1}, ...,\ \alpha_1)$. We will define $\hat{P}_n$ as the mirror inversion of all points of the set P_n. The set P_n is called odd, if $|\boldsymbol{\alpha}(0)|$ is odd for every point $\boldsymbol{\alpha} \in P_n$. The set P_n is called b-saturated, if $X(\boldsymbol{\alpha}) \subseteq P_n$ for every point $\boldsymbol{\alpha} \in P_n$, where $b = 1, 2$. We will define the concatenation of the points of sets in the following way. Let $A \subset AG(n,\ 3)$ and $B \subset AG(m,\ 3)$. Form a new set $AB \subset AG(n+m,\ 3)$ consisting of all points $\boldsymbol{\alpha} = (\alpha_1, ...,\ \alpha_n,\ \alpha_{n+1}, ...,\ \alpha_{n+m})$, where $\boldsymbol{\alpha}^{(1)} = (\alpha_1, ...,\ \alpha_n) \in A$ and $\boldsymbol{\alpha}^{(2)} = (\alpha_{n+1}, ...,\ \alpha_{n+m}) \in B$. Similarly, one can define the concatenation of the points for any number of sets.

Theorem 1. The complete set $A \subseteq AG(n,\ 3)$ is a b-saturated P_n set if and only if it satisfies the following two conditions:
$i)$ for any two distinct points $\boldsymbol{\alpha},\ \boldsymbol{\beta} \in A$, there exists such i that $\alpha_i = \beta_i = 0$, where $1 \leqslant i \leqslant n$,
$iii)$ for any triple of distinct points $\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma} \in A,\ \boldsymbol{\alpha}(0) = \boldsymbol{\beta}(0) = \boldsymbol{\gamma}(0)$ or for two of them, say for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, there exists such i that $\alpha_i = \beta_i = 0$ and $\boldsymbol{\gamma}_i \neq 0$, where $1 \leqslant i \leqslant n$.

Theorem 2. If P_n is b-saturated, complete and odd, then $C_n = P_n \cup B'_n (C_n = P_n \cup B''_n)$ is a complete cap.

Corollary 1. $c_{6,3} \geq 112$ [7, 10].

Corollary 2. $c_{11,3} \geq 5504$.

It seems that this lower bound currently is the best one. Put $E_{2n} = \{\boldsymbol{e}_{1,1} = (1,\ 0, ...,\ 0),\ \boldsymbol{e}_{1,2} = (2,\ 0, ...,\ 0),\ \boldsymbol{e}_{2,1} = (0,\ 1, ...,\ 0),\ \boldsymbol{e}_{2,2} = (0,\ 2, ...,\ 0), ...,\ \boldsymbol{e}_{2n,1} = (0,\ 0, ...,\ 1),\ \boldsymbol{e}_{2n,2} = (0,\ 0, ...,\ 2)\}$.

Theorem 3. Let P_{2n} be a b-saturated, complete and odd set. If $|\boldsymbol{\alpha}(0)| \geq 3$ for every point $\boldsymbol{\alpha} \in P_{2n}$ and $P_{2n} \cap \hat{P}_{2n} = \emptyset$ then $C_{2n+1} = \{P_{2n} \cup B'_{2n}\}\{(0)\} \cup \{\hat{P}_{2n} \cup B'_{2n}\}\{(1)\} \cup \{E_{2n}\}\{(2)\}$ is a cap.

Corollary 3. $c_{7,3} \geq 236$ [10].

For the given three complete sets $P_{n_1},\ P_{n_2}$ and P_{n_3}, form the following set $P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3}$. It is known [11, 12] that the formed set is a complete P_n-set, where $n = \sum_{i=1}^{3} n_i$ and $n_1,\ n_2,\ n_3$ are any integers.

Theorem 4. Let $P_{n_1},\ P_{n_2},\ P_{n_3}$ be b-saturated and complete sets. If two of the sets $P_{n_1},\ P_{n_2},\ P_{n_3}$, say $P_{n_1},\ P_{n_2}$, are odd, then $C_n = P_n \cup B'_{n_1}B'_{n_2}B_{n_3}$ is a complete cap, where $P_n = P_{n_1}P_{n_2}B_{n_3} \cup P_{n_1}B_{n_2}P_{n_3} \cup B_{n_1}P_{n_2}P_{n_3},\ n = \sum_{i=1}^{3} n_i$ and $n_1,\ n_2,\ n_3$ are any integers.

Corollary 4. $c_{10,3} \geq 2240$ [9].

For the given complete sets $P_{n_1},\ P_{n_2},\ P_{n_3},\ P_{n_4},\ P_{n_5}$ and P_{n_6}, form the following ten sets:

$$\begin{aligned}
A_1 &= P_{n_1}P_{n_2}P_{n_3}B_{n_4}B_{n_5}B_{n_6},\\
A_2 &= P_{n_1}P_{n_2}B_{n_3}B_{n_4}B_{n_5}P_{n_6},\\
A_3 &= P_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}B_{n_6},\\
A_4 &= B_{n_1}P_{n_2}P_{n_3}P_{n_4}B_{n_5}B_{n_6},\\
A_5 &= B_{n_1}B_{n_2}P_{n_3}P_{n_4}B_{n_5}P_{n_6},\\
A_6 &= B_{n_1}B_{n_2}P_{n_3}B_{n_4}P_{n_5}P_{n_6},\\
A_7 &= B_{n_1}P_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6},\\
A_8 &= B_{n_1}P_{n_2}B_{n_3}B_{n_4}P_{n_5}P_{n_6},\\
A_9 &= P_{n_1}B_{n_2}B_{n_3}P_{n_4}B_{n_5}P_{n_6},\\
A_{10} &= P_{n_1}B_{n_2}B_{n_3}P_{n_4}P_{n_5}B_{n_6}.
\end{aligned}$$

It is known [12, 13] that $\cup_{i=1}^{10} A_i$ is a complete P_n-set, where $n = \sum_{i=1}^{6} n_i$ and $n_1,\ n_2,\ n_3,\ n_4,\ n_5,\ n_6$ are any integers.

Theorem 5. Let $P_{n_1},\ P_{n_2},\ P_{n_3},\ P_{n_4},\ P_{n_5}$ and P_{n_6} be b-saturated and complete sets. If the sets $P_{n_1},\ P_{n_2}$ and P_{n_3} are odd, then $C_n = P_n \cup B'_{n_1}B'_{n_2}B'_{n_3}B_{n_4}B_{n_5}B_{n_6}$ is a complete cap, where $P_n = \cup_{i=1}^{10} A_i,\ n = \sum_{i=1}^{6} n_i$ and $n_1,\ n_2,\ n_3,\ n_4,\ n_5,\ n_6$ are any integers.

To construct large b-saturated P_n-sets we need the following combinatorial problem.

Problem. Let $S = \{a_1,\ a_2, ...,\ a_n\}$ be a finite set and $A \subseteq S$ is any subset of the set S. Assume that the weight of the subset $A \subseteq S$ is $2^{n-|A|}$ and denote by $w(A)$. Find such a set $B = \{A_1,\ A_2, ...,\ A_m\}$ of the subsets of S satisfying the following three conditions:

1) for any two distinct subsets $A_i,\ A_j \in B,\ A_i \cap A_j \neq \emptyset$, where $1 \leq i, j \leq m$,
2) for any triple of distinct subsets $A_i,\ A_j,\ A_k \in B$, there is an element $a_l \in S$, such that a_l belongs to two of them, say $a_l \in A_i,\ a_l \in A_j$, but $a_l \notin A_k$, where $1 \leq i, j, k \leq m,\ 1 \leq l \leq n$,
3) $\sum_1^m w(A_i)$ is maximal.

Assume that the set $B = \{A_1,\ A_2, ...,\ A_m\}$ satisfies the above three conditions. To construct a P_n-set, each subset A_j can be matched to the set of the points $X_j \subseteq AG(n,\ 3)$ in the following way: if $a_i \in A_j$ then the i'-th coordinate of all points of X_j takes 0 value, otherwise b, where $1 \leq j \leq m$, $1 \leq i \leq n$ and $b = 1, 2$. It is obvious that $\cup_1^m X_j$ is a P_n-set.

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