An Upper Bound on the Edge-Chromatic Sum of Fibonacci Cubes

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Abstract—Fibonacci cube is an isometric subgraph of the
n-dimensional cube. A proper edge-coloring of a graph G is a
mapping α : E(G) → N such that α(e) ≠ α(e′) for every pair
of adjacent edges e and e′ in G. The edge-chromatic sum of a
graph G is the minimum sum of all colors in the graph among
all its proper edge-colorings. This paper provides an upper
bound on the edge-chromatic sum of Fibonacci cubes.

Keywords— Edge-chromatic sum, Fibonacci cubes, sum edge-
coloring.

I. INTRODUCTION

Let B = {0, 1} and for n ≥ 1 set B_n = {b_1b_2...b_n | b_i ∈ B, 1 ≤ i ≤ n}. The n-cube Q_n graph is the graph defined
on the vertex set B_n, vertices b_1b_2...b_n and b_1′b_2...b_n being adjacent if b_i ≠ b_i′ holds for exactly one i ∈ {1, 2, ..., n}.

Fibonacci cubes are introduced as follows: for
n ≥ 1, let F_n = {b_1b_2...b_n ∈ B_n | b_i · b_{i+1} = 0, 1 ≤ i ≤ n − 1}. The Fibonacci cube Γ_n is the subgraph of Q_n induced by
the vertex set F_n [1].

A proper vertex-coloring of a graph G is a mapping
α : V(G) → N such that α(u) ≠ α(v) for every uv ∈ E(G). In that case α(v) is called the color of the vertex v. The vertex-
chromatic sum Σ(G) of a graph G is the minimum sum of colors of all vertices among all proper vertex-colorings of G. The concept of vertex-chromatic sum was introduced by Kubicka [2] and Supowit [3]. The problem of finding
the vertex-chromatic sum is shown to be NP-complete in general
dynamic programming algorithm for partial k-trees. In papers
[6], [7], [8], [9], [10], some approximation algorithms were
given for various classes of graphs. Some bounds for the
vertex-chromatic sum of a graph were given in [11].

A proper edge-coloring of a graph G is a mapping α : E(G) → N such that α(e) ≠ α(e′) for every adjacent e and e′. In that case α(e) is called a color of the edge e. Similar to
the vertex-chromatic sum of graphs, in [6], [12], and [13],
edge-chromatic sum of graphs was introduced. Namely, the
edge-chromatic sum Σ′(G) of a graph G is the minimum sum of
all colors in the graph among all its proper edge-colorings. In [6], Bar-Noy et al. proved that the problem of finding the
edge-chromatic sum is NP-hard for multigraphs. Later, in [12], it was shown that the problem is NP-complete for bipartite
graphs with maximum degree 3. Petrosyan and Kamalian [14]
proved that the problem is NP-complete for even more specific
class of graphs from the latter and found an 11/8-approximation algorithm for r-regular graphs. In [15], Salavati-pour
proved that determining the edge-chromatic sum is NP-complete for
r-regular graphs with r ≥ 3. The problem can be solved in
polynomial time for trees [12].

The terms and concepts that we do not define can be found
in [16].

In the present paper, we obtain an upper bound on the edge-
chromatic sum of Fibonacci cubes.

II. MAIN RESULT

Theorem 1. For any n ∈ N, we have

Σ′(Γ_n) ≤

\left( \frac{5 + 3\sqrt{5}}{100} n^2 + \frac{11 + 9\sqrt{5}}{100} n + \frac{6\sqrt{5}}{125} \right) \left( \frac{1 + \sqrt{5}}{2} \right)^n +

\left( \frac{5 - 3\sqrt{5}}{100} n^2 + \frac{11 - 9\sqrt{5}}{100} n - \frac{6\sqrt{5}}{125} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^n .

Proof. We will construct a corresponding proper edge-
coloring α_n for each Γ_n by induction on n. We denote by
Σ′(α_n) the sum of colors of all edges for the coloring α_n.
From the relation Σ′(Γ_n) ≤ Σ′(α_n), which implies from the
definition of the edge-chromatic sum, we will derive the result.

It is easy to construct α_3 and α_2 separately, so they have,
respectively, 1 and 3 as their sums.

Now let us construct the coloring α_n for n ≥ 3 assuming
that we have already constructed all α_k-s for 1 ≤ k < n. It is
known that it is possible to decompose Γ_n into two subgraphs
Γ_{n−1} and Γ_{n−2} in such a way that V(Γ_n) = V(Γ_{n−2}) ∪ V(Γ_{n−1}) and E(Γ_n) = E(Γ_{n−2}) ∪ E(Γ_{n−1}) ∪ M, where M is
a matching of 2|V(Γ_{n−2})| vertices [1]. Let us color the edges
of the matching with the color 1. For the remaining edges let
us use the corresponding colors in the colorings α_{n−2} and
α_{n−1}, and color the edge e with α_{n−2}(e) + 1 if e ∈ E(Γ_{n−2})
and α_{n−1}(e) + 1 if e ∈ E(Γ_{n−1}).

Clearly, we constructed a proper edge-coloring. Moreover,
Σ′(α_n) = |E(Γ_n)| = Σ′(α_{n−1}) + Σ′(α_{n−2}). By [1], we have that |E(Γ_n)| = \frac{F_{n+1} + 2(n+1)F_n}{8}, where F_n is the n-th
Fibonacci number.

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If we denote $\Sigma'(a_n)$ by $a(n)$, then to obtain the required inequality, it is necessary to solve the 2nd order non-homogeneous recurrence relation $a(n) = a(n-1) + a(n-2) + \frac{nF_{n+1} + 2(n+1)F_n}{F_{n+2}}$, with initial conditions $a(1) = 1$ and $a(2) = 3$. To do that, we represent $a(n)$ as the sum $a_h(n) + a_p(n)$, where $a_h(n)$ is the solution of the associated homogeneous recurrence relation, and $a_p(n)$ is the particular solution. The characteristic equation of the homogeneous relation is $\lambda^2 - \lambda - 1$, roots of which are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Therefore, $a_h(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. As for the particular solution, considering the formula of the common term of the Fibonacci sequence, we get that $a_p(n)$ has the following form: $(An^2 + Bn + C) \left(\frac{1+\sqrt{5}}{2}\right)^n + (Dn^2 + En + F) \left(\frac{1-\sqrt{5}}{2}\right)^n$. Thus, we obtained the forms of $a_h(n)$ and $a_p(n)$, and to get the formula of $a(n)$ we need to put those results in the recurrence relation, and obtain the values of the coefficients using the initial conditions.

The proper edge-colorings $\alpha_3$ and $\alpha_4$ are illustrated in Figure 1.

![Diagram](image)

**Figure 1**

**REFERENCES**


