Large Cycles in Graphs Around Conjectures of Bondy and Jung - Modifications and Sharpness

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***Abstract*—A number of new sufficient conditions for generalized cycles (large cycles including Hamilton and dominating cycles as special cases) in graphs and new lower bounds for the circumference (the length of a longest cycle) are presented inspiring a number of modifications of famous conjectures of Bondy (1980) and Jung (2001). All results (both old and new) are best possible in a sense based on three types of sharpness indicating the intervals in which the result is sharp and the intervals in which the result can be further improved. In addition, the modifications cannot be derived directly from Bondy's and Jung's conjectures as special cases.**

***Keywords*— Hamilton cycle, dominating cycle, longest cycle, large cycle.**

I. Introduction

We consider only finite undirected graphs without loops or multiple edges [1]. The set of vertices of a graph G is denoted by $V(G)$; the set of edges - by $E(G)$. For a subset S of $V(G)$, we denote by $G-S$ the maximum subgraph of G with the vertex set $V(G)-S$. For a subgraph H of G, we use $G-H$, short for $G-V(H)$.

Let α and δ be the independence number and the minimum degree of a graph G, respectively. We define σ_k by the minimum degree sum of any k independent vertices if $\alpha \geq k$; if $\alpha < k$, we set $\sigma_k = +\infty$. In particular, we have $\sigma_1 = \delta$. A graph G is Hamiltonian if G contains a Hamilton cycle, i.e., a cycle of order $|V(G)|$.

Now let Q be a cycle in G. We say that Q is a dominating cycle in G if $V(G-Q)$ is an independent set of vertices.

For a positive integer λ, the cycle Q is called a PD_λ-cycle if each path of order at least λ has at least one vertex with Q in common. Similarly, we call the cycle Q a CD_λ-cycle if each cycle of order at least λ has at least one vertex with Q in common. Actually, a PD_λ-cycle dominates all paths of order λ in G; and a CD_λ-cycle dominates all cycles of order λ in G. In terms of PD_λ and CD_λ-cycles, Q is a Hamilton cycle if and only if either Q is a PD_1-cycle or a CD_1-cycle. Further, Q is a dominating cycle if and only if either Q is a PD_2-cycle or a CD_2-cycle.

Throughout the paper, we consider a graph G on n vertices with a minimum degree δ and a connectivity κ. Further, let C be a longest cycle in G with $c=|C|$, and let $\overline{p}$ and $\overline{c}$ denote the orders of a longest path and a longest cycle in $G-C$, respectively. In particular, C is a Hamilton cycle if and only if either $\overline{p} \leq 0$ or $\overline{c} \leq 0$. Similarly, C is a dominating cycle if and only if either $\overline{p} \leq 1$ or $\overline{c} \leq 1$.

In 1980, Bondy [2] conjectured a common generalization of some well-known degree-sum conditions for PD_λ-cycles ($(\sigma,\overline{p})$-version) including Hamilton cycles (PD_1-cycles) and dominating cycles (PD_2-cycles) as special cases.

Conjecture A (Bondy [2], 1980): $(\sigma,\overline{p})$-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\sigma_{\lambda+1} \geq n+\lambda(\lambda-1)$, then $\overline{p} \leq \lambda-1$.

The long cycle's analogue (the so called reverse version) of Bondy's conjecture (Conjecture A) can be formulated as follows.

Conjecture B: (reverse, σ, $\overline{p}$)-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{p} \geq \lambda-1$, then $c \geq \sigma_\lambda - \lambda(\lambda-2)$.

The initial motivations of Conjecture A and Conjecture B come from their minimum degree versions - the most popular and much studied versions, which also remain unsolved.

Conjecture C (Bondy [2],1980): $(\delta,\overline{p})$-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1}+\lambda-2$, then $\overline{p} \leq \lambda-1$.

Conjecture D (Jung [3], 2001): (reverse, δ, $\overline{p}$)-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{p} \geq \lambda-1$, then $c \geq \lambda(\delta-\lambda+2)$.

We consider the exact analogues of Bondy's generalized conjecture (Conjecture A) and its reverse version (Conjecture B) for CD_λ-cycles which we call $(\sigma,\overline{c})$ and $(reverse,\sigma,\overline{c})$-versions, respectively.

Conjecture E: $(\sigma,\overline{c})$-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\sigma_{\lambda+1} \geq n+\lambda(\lambda-1)$, then $\overline{c} \leq \lambda-1$.

Conjecture F: (reverse, σ, $\overline{c}$)-version.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \lambda-1$, then $c \geq \sigma_\lambda - \lambda(\lambda-2)$.

In [4] and [5], the validity of minimum degree versions of Conjectures E and F is proved with significant improvements.

Theorem A ([4], 2009): $(\delta,\overline{c})$-version, $\overline{c}$-improvement.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1}+\lambda-2$, then $\overline{c} \leq \min\{\lambda-1,\delta-\lambda\}$.

Theorem B ([5], 2022): $(\delta,\overline{c})$-version, κ-improvement.
Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda,\delta-\lambda+1\}$ and $\delta \geq \frac{n+2}{\lambda+1}+\lambda-2$, then $\overline{c} \leq \lambda-1$.

https://doi.org/10.51408/csit2023_32

Theorem C ([5], 2022): $(\delta, \overline{c})$-version, $(\overline{c}, \kappa)$-improvement.
Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Theorem D ([4], 2009): (reverse, δ, $\overline{c}$)-version, $\overline{c}$-improvement.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem E ([5], 2022): (reverse, δ, $\overline{c}$)-version, κ-improvement.
Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\overline{c} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem F ([5], 2022): (reverse, δ, $\overline{c}$)-version, $(\overline{c}, \kappa)$-improvement.
Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\overline{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

II. Results

In this paper, we present the following new sufficient conditions for large cycles (direct versions) and new lower bounds for the circumference (reverse versions) inspiring new modified versions of conjectures of Bondy and Jung. All results are best possible in a sense and cannot be derived directly from conjectures of Bondy and Jung as special cases.

Theorem 1: $(\delta, \overline{c})$-version, $\overline{c}$-modification.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \delta - \lambda$.

Theorem 2: $(\delta, \overline{c})$-version, κ-modification.
Let C be a longest cycle in a $(\delta - \lambda + 1)$-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \lambda - 1$.

Theorem 3: $(\delta, \overline{c})$-version, $(\overline{c}, \kappa)$-modification.
Let C be a longest cycle in a $(\delta - \lambda + 1)$-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \delta - \lambda$.

Theorem 4: $(\delta, \overline{c})$-version, κ-modification, $\overline{c}$-improvement.
Let C be a longest cycle in a $(\delta - \lambda + 1)$-connected $(1 \leq \lambda \leq \delta)$ graph G of order n. If $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \min\{\lambda - 1, \delta - \lambda\}$.

Theorem 5: $(\delta, \overline{c})$-version, $\overline{c}$-modification, κ-improvement.
Let C be a longest cycle in a graph G of order n and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 1\}$ and $\delta \geq \frac{n+2}{\lambda+1} + \lambda - 2$, then $\overline{c} \leq \delta - \lambda$.

Theorem 6: (reverse, δ, $\overline{c}$)-version, $\overline{c}$-modification.
Let C be a longest cycle in a λ-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem 7: (reverse, δ, $\overline{c}$)-version, κ-modification.
Let C be a longest cycle in a $(\delta - \lambda + 2)$-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \lambda - 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem 8: (reverse, δ, $\overline{c}$)-version, $(\overline{c}, \kappa)$-modification.
Let C be a longest cycle in a $(\delta - \lambda + 2)$-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem 9: (reverse, δ, $\overline{c}$)-version, κ-modification, $\overline{c}$-improvement.
Let C be a longest cycle in a $(\delta - \lambda + 2)$-connected $(1 \leq \lambda \leq \delta)$ graph G. If $\overline{c} \geq \min\{\lambda - 1, \delta - \lambda + 1\}$, then $c \geq \lambda(\delta - \lambda + 2)$.

Theorem 10: (reverse, δ, $\overline{c}$)-version, κ-improvement, $\overline{c}$-modification.
Let C be a longest cycle in a graph G and λ a positive integer with $1 \leq \lambda \leq \delta$. If $\kappa \geq \min\{\lambda, \delta - \lambda + 2\}$ and $\overline{c} \geq \delta - \lambda + 1$, then $c \geq \lambda(\delta - \lambda + 2)$.

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