On Interval Coloring Thickness of Some Classes of Bipartite Graphs

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***Abstract*—A proper edge-coloring of a graph G is a mapping $\alpha : E(G) \longrightarrow \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges e and e' in G. An interval edge-coloring of a graph G is a proper edge-coloring satisfying that the colors on the edges incident with any vertex form an interval of integers. A graph G is interval colorable if it has an interval edge-coloring with some number of colors. It is well known that not all graphs are interval colorable; a simple example is K_3. Recently, Asratian, Casselgren and Petrosyan introduced and studied a new notion, the interval coloring thickness of a graph G, denoted by $\theta_{int}(G)$, which is the minimum number of interval colorable edge-disjoint subgraphs of G the union of which is G. In particular, they proved that for any graph G on n vertices, $\theta_{int}(G) \leq 2\lceil \frac{n}{5} \rceil$. Using a probabilistic method, Axenovich and Zheng improved this upper bound by showing that $\theta_{int}(G) = o(n)$ for any graph G on n vertices. In this paper, we study the interval coloring thickness of some classes of bipartite graphs. In particular, we determine or bound the interval coloring thickness of bipartite graphs constructed based on finite affine and projective planes.**

***Keywords*— Interval edge-coloring, interval coloring thickness, bipartite graph.**

I. Introduction

We use [1], [2] for terminology and notation not defined here. We consider graphs that are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or $d(v)$), the maximum degree of G by $\Delta(G)$ and the chromatic index of G by $\chi'(G)$.

The classical concept of thickness of a graph G was introduced by Harary [3] in 1961 as a measure, which shows how far a graph G is from being planar. Formally, the thickness $\theta(G)$ of a graph G is the minimum number of planar subgraphs the union of which is G. Clearly, $\theta(G) = 1$ if and only if G is planar. In [4], [5], Beineke, Harary and Moon investigated the thickness of complete and complete bipartite graphs. In [6], Kleinert determined the thickness of hypercubes. Generally, it is NP-complete to decide whether a graph has thickness two [7]. There are many papers devoted to this topic, in particular, survey on the topic can be found in [8].

An interval t-coloring of a graph G is a proper edge-coloring α of G with colors $1, \dots, t$ such that all colors are used and for each $v \in V(G)$, the set of colors of the edges incident to v is an interval of integers. A graph G is interval colorable if G has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of interval edge-colorings was introduced by Asratian and Kamalian [9] (available in English as [10]) in 1987 and was motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. This problem corresponds to the problem of finding an interval edge-coloring of a bipartite multigraph. In [9], [10], Asratian and Kamalian observed that if G is interval colorable, then $\chi'(G) = \Delta(G)$. Moreover, they also proved [9], [10] that if a triangle-free graph G has an interval t-coloring, then $t \leq |V(G)| - 1$. In [16], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved that the complete bipartite graph $K_{m,n}$ has an interval t-coloring if and only if $m + n - \gcd(m, n) \leq t \leq m + n - 1$, where $\gcd(m, n)$ is the greatest common divisor of m and n. In [11], [12], Petrosyan, Khachatrian and Tananyan investigated interval colorings of complete graphs and n-dimensional cubes. In particular, they proved that the n-dimensional cube Q_n has an interval t-coloring if and only if $n \leq t \leq \frac{n(n+1)}{2}$. Generally, it is an NP-complete problem to determine whether a bipartite graph has an interval coloring [13]. In fact, for every positive integer $\Delta \geq 11$, there exists a bipartite graph with a maximum degree Δ that has no interval coloring [14]. Nevertheless, some classes of graphs have been proved to admit interval colorings; it is known, for example, that trees, regular and complete bipartite graphs [9], [15], [16], subcubic graphs with $\chi'(G) = \Delta(G)$ [17], doubly convex bipartite graphs [18], [19], grids [20], outerplanar bipartite graphs [21], $(2, b)$-biregular graphs [15], [22], [23] and $(3, 6)$-biregular graphs [24] have interval colorings, where an (a, b)-biregular graph is a bipartite graph where the vertices in one part all have a degree a and the vertices in the other part all have a degree b.

The concept of interval coloring thickness $\theta_{int}(G)$ of a graph G was recently introduced by Asratian, Casselgren and Petrosyan [17] as a synthesis of two previous concepts. Formally, the interval coloring thickness $\theta_{int}(G)$ of a graph G is the minimum number of interval colorable edge-disjoint subgraphs of G the union of which is G. The problem of determining of the interval coloring thickness of a graph was motivated by some problems from scheduling theory; here, we mention one of them. Suppose that some firms organize job interviews for possible candidates during a couple of days. We need to provide the schedule of job interviews where neither firm representatives nor candidates wait between their meetings during these days. If we construct a bipartite graph G with parts F and C, where vertices in F represent firms and vertices in C represent candidates, and edges represent the required interviews, then the minimum number of days needed for a schedule of job interviews without waiting periods is precisely equal to $\theta_{int}(G)$.

In [17], Asratian, Casselgren and Petrosyan proved a general upper bound on $\theta_{int}(G)$ for an arbitrary graph G in terms of its chromatic index. In particular, they showed that for any graph G, $\theta_{int}(G) \leq 2\left\lceil \frac{\chi'(G)}{5} \right\rceil$. They also derived some upper bounds on $\theta_{int}(G)$ for bipartite graphs. In particular, it was shown that for any bipartite graph G, $\theta_{int}(G) \leq \left\lceil \frac{\Delta(G)}{3} \right\rceil$, and if G is also Eulerian, then $\theta_{int}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil$. Recently, Axenovich and Zheng [25] improved the general upper bound by showing that $\theta_{int}(G) = o(n)$ for any graph G on n vertices. They also constructed bipartite graphs whose possible number of colors in interval colorings has arbitrarily many large gaps. In [17], Asratian, Casselgren and Petrosyan suggested the following natural problem: for any positive integer k, is there a graph G with $\theta_{int}(G) = k$? Using a probabilistic method, Axenovich, Girão, Hollom, Portier, Powierski, Savery, Tamitegama and Versteegen [26] gave a positive answer to this problem.

In this paper, we study interval coloring thickness of bipartite graphs constructed based on finite affine and projective planes.

II. Main Results

For a bipartite graph G with bipartition (X, Y), we denote the maximum degree of the vertices in X (Y) by $\Delta(X)$ ($\Delta(Y)$). Before we begin our considerations we need the following result from [17].

Theorem 1: If G is a bipartite graph with bipartition (X, Y), then

$$\theta_{int}(G) \leq \min\{\Delta(X), \Delta(Y)\}.$$

For $n, k \in \mathbb{N}$ ($k \geq 2$), define a bipartite graph $P(n,k)$ with bipartition (X, Y) as follows: $V(P(n,k)) = X \cup Y$, where

$$X = \{a\} \cup \left\{c_{(i,j)} |\ 1 \leq i < j \leq k\right\},$$

$$Y = \left\{b_l^{(i)} |\ 1 \leq i \leq k, 1 \leq l \leq n\right\} \text{ and}$$

$$E(P(n,k) = \left\{ab_l^{(i)} |\ 1 \leq i \leq k, 1 \leq l \leq n\right\} \cup$$

$$\cup \left\{b_l^{(i)}c_{(i,j)}, b_l^{(j)}c_{(i,j)} |\ 1 \leq i < j \leq k, 1 \leq l \leq n\right\}.$$

Clearly, $|V(P(n,k))| = nk + 1 + \binom{k}{2}$, $|E(P(n,k))| = nk + 2n\binom{k}{2}$ and $d(a) = nk$, $d\left(b_l^{(i)}\right) = k$, $d(c_{(i,j)}) = 2n$ (See Fig. 1). Theorem 1 implies that $\theta_{int}(P(n,k)) \leq k$. Here, we determine the interval coloring thickness of these graphs.

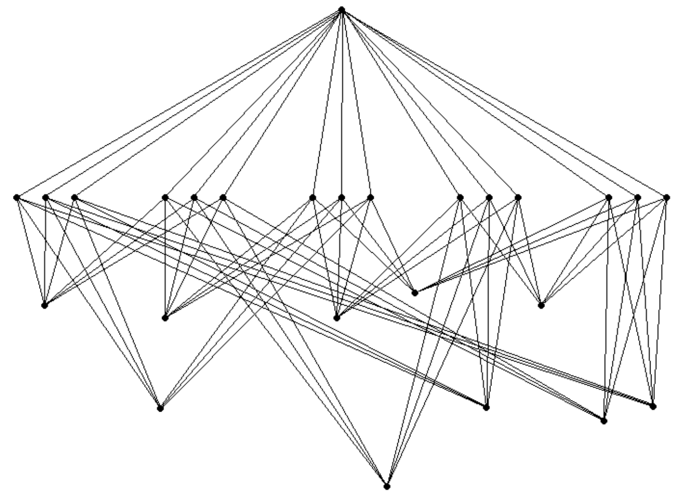

Fig. 1. A bipartite graph P(3,5).

Theorem 2: If $n + k - 1 < \frac{nk}{2}$ ($n, k \in \mathbb{N}$), then $\theta_{int}(P(n,k)) = 2$.

Using finite projective planes, Petrosyan and Khachatrian [14] constructed a family of bipartite graphs that do not admit interval colorings. This family of graphs generalizes a construction first given by Erdős [27] in 1991. Since the interval coloring thickness of graphs from this family is unknown, in [17], Asratian, Casselgren and Petrosyan suggested the problem of determining the interval coloring thickness of Erdős family of graphs as an open problem (Problem 4.2 from [17]). Here, we present some progress on this problem.

Let $\pi(n)$ be a finite projective plane of order $n \geq 2$, $P = \{1, 2, \ldots, n^2 + n + 1\}$ be the set of points and $L = \{l_1, l_2, \ldots, l_{n^2+n+1}\}$ the set of lines of $\pi(n)$. Let $A_i = \{k \in l_i |\ 1 \leq k \leq n^2 + n + 1\}$ for every $1 \leq i \leq n^2 + n + 1$; then $|A_i| = n + 1$ for every i, and $A_i \neq A_j$ if $i \neq j$. For a sequence of $n^2 + n + 1$ integers $r_1, r_2, \ldots, r_{n^2+n+1} \in \mathbb{N}$ ($r_1 \geq \ldots \geq r_{n^2+n+1} \geq 1$), we define the graph $Erd(r_1, \ldots, r_{n^2+n+1})$ as follows:

$$V(Erd(r_1, \ldots, r_{n^2+n+1})) = \{u\} \cup \{1, \ldots, n^2 + n + 1\} \cup \left\{v_1^{(l_i)}, \ldots, v_{r_i}^{(l_i)} |\ 1 \leq i \leq n^2 + n + 1\right\},$$

$$E(Erd(r_1, \ldots, r_{n^2+n+1})) =$$

$$\bigcup_{i=1}^{n^2+n+1} \left(\{uv_1^{(l_i)}, \ldots, uv_{r_i}^{(l_i)}\} \cup \{v_1^{(l_i)}k, \ldots, v_{r_i}^{(l_i)}k |\ k \in A_i\}\right).$$

The graph $Erd(r_1, r_2, \ldots, r_{n^2+n+1})$ is a connected bipartite graph with $n^2 + n + 2 + \sum_{i=1}^{n^2+n+1} r_i$ vertices and maximum

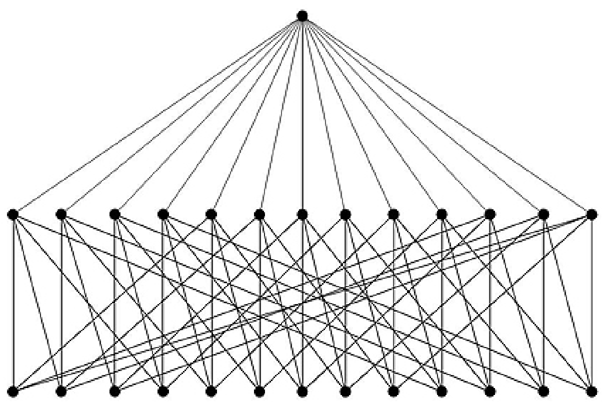

Fig. 2. A bipartite graph Erd(1,1,1,1,1,1,1,1,1,1,1,1,1).

degree $\sum_{i=1}^{n^2+n+1} r_i$ (See Fig. 2). The following result was proved by Petrosyan and Khachatrian [14].

Theorem 3: If $\sum_{i=n+2}^{n^2+n+1} r_i > 2(n+1)$, then $Erd(r_1, \ldots, r_{n^2+n+1}) \notin \mathfrak{N}$.

Theorem 1 implies that

$$\theta_{int}(Erd(r_1, \ldots, r_{n^2+n+1}) \leq n+2.$$

If $r_1 = r_2 = \cdots = r_{n^2+n+1} = r$, then the graph $Erd(r_1, r_2, \ldots, r_{n^2+n+1})$ we shortly denote by $Erd_n(r)$. Now we are able to formulate our next result.

Theorem 4: For any $n, r \in \mathbb{N}$, we have

1) if $n, r \geq 2$, then $\theta_{int}(Erd_n(r)) = 2$,
2) if $r = 1$ and $n \geq 3$, then $\theta_{int}(Erd_n(r)) = 2$.

Let $\alpha(n)$ be a finite affine plane of order $n \geq 2$, $P = \{1, 2, \ldots, n^2\}$ be the set of points and $L = \{l_1, l_2, \ldots, l_{n^2+n}\}$ the set of lines of $\alpha(n)$. Define the bipartite graph $A(n)$ with bipartition (P, L) as follows:

$$V(A(n)) = P \cup L, E(A(n)) = \{pl|\ p \in P, l \in L, p \in l\}.$$

Our last result concerns an upper bound on $\theta_{int}(A(n))$.

Theorem 5: For any positive integer $n \geq 2$, $\theta_{int}(A(n)) \leq 2$. Moreover, $\theta_{int}(A(2)) = \theta_{int}(A(3)) = 1$.

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