Properties of Increasing Hazard Rate of Waiting Times in the $GI|G|1|\infty$ Queueing Model

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Abstract— In the presented article, we consider the queuing model GI|G|1|∞ with the FIFO service discipline, and the service time of the n- th call is denoted by ν_n, n≥ 1. It is assumed that the sequences of RV's, where $u_n = t_n - t_{n-1}$, n≥ 1 and $\{\nu_n\}$, n≥1 are sequences of independent, identically distributed RV's with DF A(t), A(+0)=0 and B(t), B(+0)=0, respectively. The queuing model is investigated under the condition of model loading $\rho_1 = \frac{\beta_1}{\alpha_1}$<1, where α_1=Eu_n and β_1=Eν_n for all n≥ 1 and IHR DF B(x), which is equivalent to not increasing by the relation $\bar{B}$(x+t)/$\bar{B}$(t), where we denote IHR DF $\bar{B}$(t)= 1-B(t) for t∈[0, + ∞). Customs are numbered in the order of receipt by the numbers 1,2, ..., i.e. At the moment t=0, there are no calls in the system.

Denote by $w_n, n \geq 1$ the waiting time until the service beginning of $n - th$ customer and by w(t), t> 0 − the virtual waiting time starting at the moment t, or more precisely: the length of the time interval starting at the moment t and ending at the time when the system is free from customers that arrived in the system before time t.

It is known that the following stationary limits exist:

$$w_n \Rightarrow w, n \to +\infty \text{ and } w(t) \Rightarrow w^*, t \to +\infty, \quad (1)$$

where ⇒ denotes the sign of weak convergence (see [4], p.68 and p. 139).

Denote $\alpha_1 = Eu_1$ and $\beta_1 = Ev_1$, where E is a sign of mathematical expectation, and assume that

$$0<\alpha_1<+\infty \text{ and } 0<\beta_1<+\infty.$$

Due to Theorem 2.2 (see [4], p.73) and Theorem 3.2 (see [4], p.139), it follows that under the condition,

$$\rho_1 = \frac{\beta_1}{\alpha_1}<1 \text{ } (\rho_1 \text{ is the traffic intensity of the system}) \quad (2)$$

the DF's $W(x) = P(w \leq x)$ and $W^*(x) = P(w^* \leq x)$, are proper DFs, i.e.,

$$W(+\infty) = W^*(+\infty) = 1.$$

In the present paper, the queueing model GI|G|1| ∞ in stationary conditions is considered. For waiting time w_n with $DF\,W(x), n \geq 1$, and the virtual waiting time at the moment t $w(t)$ exist $w_n \Rightarrow w, n \to +\infty$ and $w(t) \Rightarrow w^*, t \to +\infty$, where ⇒ denotes the sign of weak convergence.

In the paper, the model with FIFO discipline and $\rho_1 = \frac{\beta_1}{\alpha_1}$<1 (see (2)) and DF B(x) of customers service times is IHR DF. Then both random variables w and w* are IHR.

Keywords—increasing hazard rate (IHR), the k-th ladder point, the k-th ladder height, virtual waiting time, Takac's, Cohen's and Hooke's formulas.

I. Problem's formulation

Queue models form is a special class of mathematical models that describes the behavior of a huge variety of complex systems. Their theoretical analysis is necessary at the stage of designing such complex systems as information networks, in particular, Internet networks, automated computing systems, supply systems, transport complexes, medical care, etc.

In the case of a common incoming flow and a common distribution of service times, a mathematical theory has now taken shape for a structurally simple model GI|G|1|∞ with one flow, one server, with waiting and service in the order of receipt. It should also be noted that there are many results on multichannel queuing models, but it is still far from building an exact theory. The mathematical theory of the model GI|G|1|∞ under the first-come-first-served discipline admits a rigorous sequential presentation. It is based on the theory of renewal, random walks, Markov processes, processes with independent increments, and combinatorial methods.

Exact methods of analysis with the complexity of the structures of queue models with non-Poisson incoming flows and with a common service time distribution function, as a rule, do not lead to the desired results. Now they give way to asymptotic methods.

II. Description of GI|G|1|∞ model

One of the important notions in the mathematical theory of reliability is the increasing hazard rate (IHR) properties of elements, which form reliability systems.

Definition 1 (see [1], page 41). *Random variable (RV)* $\xi \geq 0$ *and its distribution function (DF)*

$$F(x) = P(\xi \leq x),$$

where F - satisfying conditions:$F(+0) < 1$, $F(t) < 1$ *for* $t \in R^+ = (0, +\infty)$, $F(+\infty) = 1$, *are referred to as IHR RV and IHR DF, respectively, if for* $x \in R^+$ *the ratio* $\bar{F}(x+t)/\bar{F}(t)$, *where* $\bar{F}(t) = 1 - F(t)$ *for* $t \in [0, +\infty)$ *(1)*
is non-decreasing with respect to t.

Here P denotes the sign of probability and, obviously, condition (1) is equivalent to the non – increase with respect to t for the ratio $\bar{F}(x+t)/\bar{F}(t)$, where $\bar{F}(t) = 1 - F(t)$ for $t \in [0, +\infty)$.

Consider the following GI|G|1|∞ model. The customers arrive at the random moments $t_1, t_2, ...$ in the single server

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queue, where $0 < t_1 \le t_2 \le \cdots$. We enumerate customers in order of their arrivals by numbers $1,2,\ldots, i.e.,$ the $n-th$ customer arrives at the moments $t_n, n \ge 1$.

Denote by $v_n, n \ge 1$ the duration of the $n-th$ customer's service time. The customers are served in accordance with FIFO (first in – first out) discipline. At the moment $t = 0$, there are no customers in the system. Assume that the sequences of random variables (RV's) $\{u_n\}$, where $u_n = t_n - t_{n-1}, n \ge 1,$ and $\{v_n\}$ are independent and form sequences of independent, identically distributed (IID) RV's with DF's A(t), A(+0)=0 and B(t), B(+0)=0, respectively. Note that, the assumptions A(+0)=0 and B(+0)=0 are technical. In particular, condition A(+0)=0 implies expression $P(0 < t_1 < t_2 < \cdots) = 1$, (see [4], p.10).

Denote by $w_n, n \ge 1$ the waiting time until the service beginning of $n-th$ customer and by w(t), t> 0 – the virtual waiting time starting at moment t, or more precisely: the length of the time interval starting at moment t and ending at the time when the system is free from customers that arrived in the system before time t.

It is known that the following stationary limits exist:

$$w_n \Rightarrow w, n \to +\infty \text{ and } w(t) \Rightarrow w^*, t \to +\infty,$$

where ⇒ denotes the sign of weak convergence (see [4], p.68 and p. 139).

Denote $\alpha_1 = Eu_1$ and $\beta_1 = Ev_1$, where E is a sign of mathematical expectation, and assume that

$0<\alpha_1<+\infty$ and $0<\beta_1<+\infty$.

Due to Theorem 2.2 (see [4], p.73) and Theorem 3.2 (see [4], p.139), it follows that under the condition

$$\rho_1 = \frac{\beta_1}{\alpha_1} < 1 \text{ } (\rho_1 \text{ is the traffic intensity of the system}) \tag{2}$$

the DF's $W(x) = P(w \le x)$ and $W^*(x) = P(w^* \le x)$ are proper DF's, i.e.,

$$W(+\infty) = W^*(+\infty) = 1.$$

They are called waiting times for stationary DF's.

Note the following important result (see Theorem 3.6, [4], p.161-162).

In the model GI|G|1|∞ with FIFO discipline and $\rho_1 < 1$ for the validity of equality $W = W^*$, the form

$$A(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{\alpha_1}\right), if\ x \ge 0, \\ 0, if\ x < 0. \end{cases}$$

is necessary and sufficient.

Thus, under the FIFO discipline in the frame of *the* GI|G|1|∞ model, the stationary waiting times coincide only and only in the case of M|G|1|∞ model.

That is why the main result of article [2] for the M|G|1|∞ model substantiates the following formulation of the problem.

Problem. Find conditions on DF's *A(x)* and *B(x)* such that the DF's W and W^* are *IHR DF's*.

The solution to this problem is the aim of the present article.

III. STRUCTURE OF RV

For $n \ge 1$ denote: $X_n = v_n - u_{n+1}$ and $S_n = X_1 + X_2 + \cdots + X_n, S_0 = 0$.

Due to (2.1.11) ([4], p. 68), the RV's W and $\sup_{n\ge 0} S_n$ are identically distributed. That is why

$$p_0 = P(w = 0) = P(S_n \le 0, n \ge 1) \tag{3}$$

Moreover, the condition (1) is equivalent to inequalities (see [4], p.85)

$$0 < p_0 < 1. \tag{4}$$

Remark 1. For $\rho_1 < 1$ denote

$$p_0^* = P(w^* = 0). \tag{5}$$

Then:

1) According to [5], p. 100 (see (1) and (5))

$$0 < p_0^* = 1 - \rho_1 < 1; \tag{6}$$

2) According to [5], p. 100 (see also (1), (3) and (5))

$$p_0 = p_0^* \text{ } (=1-\rho_1). \tag{7}$$

if and only if (iff) the DF A(x) takes the form (2), i.e., in the case of model M|G|1|∞.

Definition 2 ([6], Ch. XII, Sect. 1, p. 458). For sequence $\{S_n\}_0^\infty$, *the point* $(n, S_n), n \ge 1$ *is called a ladder point, if* S_n *exceeds the values* $S_0(=0), S_1, S_2 \ldots, S_{n-1}$. By *definition, the first ladder point (if exists), if n is the first index for which* $S_n > 0$, *i.e.,* $(\tau_1, \zeta_1) = (n, S_n)$.

By definition, for any entire number k ≥ 1 the k-th ladder point (τ_k, ζ_k) (if exists) is defined by equalities

$$\tau_\kappa \stackrel{d}{=} \nu_1 + \nu_2 + \ldots + \nu_\kappa, \quad \zeta_\kappa \stackrel{d}{=} \xi_1 + \xi_2 + \ldots + \xi_\kappa, \tag{8}$$

where $\{\nu_n\}_{n=1}^\infty$ and $\{\xi_n\}_{n=1}^\infty$ are sequences of IID RV's being identically distributed with τ_1 and ζ_1, respectively. The sign d in (8) says that DF's of both sides of stochastic equality coincide. The equalities (8) are consequences of *Definition* 2 and of the fact that the sequence

$$S_{\tau_1} - S_{\tau_1}, S_{\tau_1+1} - S_{\tau_1}, S_{\tau_1+2} - S_{\tau_1}, \ldots$$

Present the "explicit stochastic copy" of the sequence $S_0(=0), S_1, S_2, \ldots$.

Due to *Definition 2*, we have that

$$P(\tau_1 = n) = P\left(S_k \le 0, k = \overline{1,(n-1)}, S_n > 0\right), n \ge 1. \tag{9}$$

From (9) it follows

$$\begin{aligned} \textstyle\sum_{n\ge 1} P(\tau_1 = n) &= 1 - P(S_k \le 0, k \ge 1) = \\ &= P(w = 0) = 1 - p_0 < 1. \end{aligned} \tag{10}$$

where equalities (3) and (4) were used. The RV's τ_1 *and* ξ_1 are not defined if none among the events $\{\tau_1 = n\}, n \ge 1$ takes place. That is why in our case $\rho_1 < 1$ they are non-proper RV's with the same defect p_0 (see (3) and (10). Denote by ν the random number of ladder points of sequence $\{S_n\}_{n=0}^\infty$. For any $n \ge 1$, the equalities hold:

$$\begin{aligned} P(\nu = n) &= \sum_{k\ge n} P(\nu = n, \tau_n = k) = \\ &= \sum_{k\ge n} P(\tau_n = k, S_{k+m} - S_k \le 0, m \ge 0) = \\ &= \textstyle\sum_{k\ge n} P(\tau_n = k) \cdot P(S_{k+m} - S_k \le 0, m \ge 0) = \\ &= P(S_m \le 0, m \ge 0) \cdot \sum_{k\ge n} P(\tau_n = k) = \\ &= p_0 \cdot (\sum_{k\ge 1} P(\tau_1 = k))^n = p_0 \cdot (1 - p_0), \end{aligned} \tag{11}$$

where the independence of τ_n and $S_{t_n+m} - S_{t_n}$ for any $m \ge 0$ the identical distributive of sequences $\{S_n\}_{n=0}^\infty$ and $\{S_{n+m} - S_m\}_{n=0}^\infty$ for any $m \ge 0$, and formulas (3), (8) and (10) were used.

Hence, if we additionally define

$$P(\tau_1 = 0) = P(\xi_1 = 0) = P(\nu_1 = 0) = p_0,$$

Then we get rid of the uncertainty in the definition of τ_1, ξ_1 and ν_1. Moreover, one may formulate the following statement:

Lemma 1. Let $\{\xi_n\}_{n=1}^{\infty}$-be a sequence of non-negative IID RV's with DF

$$P(\xi_1 \le x) = P(S_{\tau_1} \le x), x \in R^+, P(\xi_1 = 0) = p_0,$$

be an entire geometrical index with the parameter $(1 - p_0)$, which doesn't depend on $\{\xi_n\}_{n=1}^{\infty}$. Then for $\rho_1 < 1$ in the model GI|G|1|∞ with FIFO discipline the representation holds

$$w \stackrel{d}{=} \xi_1 + \xi_2 + \ldots + \xi_\nu . \quad (12)$$

Remark 2. Formula (12) represents the analog of Cohen's formula for stationary waiting time $w = w^*$ in the M|G|1|∞ model with discipline FIFO (see [4], p.100). In this particular case, we have

$$p_0 = 1 - \rho_1,\ P(\xi_1 \le x) = \beta^{-1} \cdot \int_0^x (1 - B(u))\, du. \quad (13)$$

IV. SIMPLIFICATION OF THE PROBLEM

The "minimal" condition for the Problem's solution is the assumption: DF B(x) of customers service times duration is IHR DF. In [2], the IHR of RV w is a particular case of the M|G|1|∞ model with discipline FIFO has been established exactly under this condition. At the same time the following statement has been proved.

Theorem 1. Let $\{\xi_n\}_{n=1}^{\infty}$-be a sequence of IID RV's with IHR DF, and $\nu \ge 0$- be an random index, which doesn't depend on $\{\xi_n\}_{n=1}^{\infty}$. Then random sum $\xi_1 + \xi_{n2} + \cdots + \xi_\nu$ is an IHR RV.

Proving that DF (13) is an IHR DF in [2], with the help of the Cohen's formula and Theorem 1, it was established that RV w is an IHR RV.

In the case of the GI|G|1|∞ model we established the conciseness of DF of RV w and DF of the last ***ladder height*** S_ν, where ν denotes the random number of ladder points of the sequence $\{S_n\}_{n=0}^{\infty}$. (Rv's $S_{\tau_k}, k \ge 1$ are called ladder heights).

Theorem 1 is applicable to (12) in our case. According to *Theorem 1* and *Lemma 1,* the Problem for w is simplified: for the establishment of IHR of the RV w , the IHR of the last ladder height, it is enough to prove the IHR of the first ladder height naturally, for already defined till proper RV ξ_1.

The following ***question*** arises: is it enough for the assumption on IHR DF B(x) for RV ξ_1 to be an IHR DF in random sum (12)? To answer this question, one needs to find "suitable" representation for DF $P(\xi_1 \le t), t \in R^+$.

Denote by $K(x)$, $x \in R^1 = (-\infty, +\infty)$ the DF of RV $X_n, n \ge 1$. Then (see, for instance, [4], p.65)

$$K(x) = \int_0^{+\infty} \big(1 - A(y - x)\big) dB(y), x\epsilon\, R^1. \quad (14)$$

Since $S_n = X_1 + X_2 + \cdots + X_n, n \ge 0, S_0 = 0$ and $\{X_n\}_{n\ge1}$ is a sequence of IID RV's , therefore for entire $n \ge 1$ *and* $x\epsilon R^1$ we get:

$$D_n(x) \stackrel{\text{def}}{=} P(S_k \le 0, r = \overline{0, n-1}, S_n \le x) =$$
$$= \int_{y_1=-\infty}^{0} dK(y_1) \int_{y_2=-\infty}^{-y_1} dK(y_2) \ldots$$
$$\ldots \int_{y_{n-1}}^{-y_1-\cdots-y_{n-2}} dK(y_{n-1}) \int_{y_n=-\infty}^{x-y_1-\cdots-y_{n-1}} dK(y_n) =$$
$$= \iint \ldots \int_{G_n(\vec{y}_n;x)} \prod_{i=1}^{n} d_{y_i} K(y_i), \quad (15)$$

where $\vec{y}_n \triangleq (y_1, y_2, \ldots, y_n)$, *DF K(x)* is defined by (14), and integration at the right-hand side of (15) is carried out in an n-dimensional area:

$$G_n(\vec{y}_n; x) = \{\vec{y}_n: y_1 \le 0, y_1 + y_2 \le 0, y_1 + \cdots + y_{n-1} \le 0, y_1 + \cdots + y_{n-1} + y_n \le x\}.$$

For n=1, we have $D_1(x) \triangleq K(x) - a$ proper DF with $x\epsilon R^1$ and for $n > 1$ DF is $D_n(x)$ a non-proper DF. One may introduce also the function

$$D_0(x) = \begin{cases} 1, \text{ if } x \ge 0, \\ 0, \text{ if } x < 0, \end{cases} \quad (16)$$

which on average coincides with the definition of D_n in (15) for n=0.

Further, for n=2 and $y\epsilon(-\infty, 0]$, we have

$$R_n(y) \stackrel{\text{def}}{=} P(S_k \le 0, k = \overline{1, n-1}, S_{n-1} - u_{n+1} \le y) =$$
$$= \int_{-\infty}^{0} D_{n-1}(y - z)\, d_z(1 - A(-z)). \quad (17)$$

Here u_{n+1} presents the random length of the interval between the arrival epochs of the $n-$th and (n+1)-st customers ($n \ge 1$) and

$$P(-u_{n+1} \le x) = P(u_{n+1} \ge -x) = \begin{cases} 1 , \text{ if } x \ge 0, \\ 1 - A(-x), \text{ if } x < 0. \end{cases}$$

It is easy to see that for $n = 1$ and $y\epsilon(-\infty, 0]$

$$R_1(y) \stackrel{\text{def}}{=} P(-u_2 \le y) = 1 - A(-y),$$

which may also be written in the form (17) with $n = 1$, taking into account (16). Finally, introduce the non-decreasing positive function

$$R(y) = \sum_{n\ge1} R_n(y) < 1, y\epsilon(-\infty, 0]. \quad (18)$$

The preliminary technical evaluations for getting a "suitable" representation for $P(\xi_1 > t)$, are completed.

Lemma 2. For $t \ge 0$, the following representation holds

$$P(\xi_1 > t) = \int_{-\infty}^{0} \bar{B}(t + |y|)\, dR(y),$$

where, for $y\epsilon(-\infty, 0]$, the non-decreasing function R(y) is defined by equalities (14) – (18).

The proof follows from the following equalities: for $t\epsilon[0, +\infty)$ we have

$$P(\xi_1 > t) = \sum_{n\ge1} P(\xi_1 > t, \tau_1 = n) =$$
$$= \sum_{n\ge1} P(S_k \le 0, k = \overline{0, n-1}, S_n > t) =$$
$$= \sum_{n\ge1} P(S_k \le 0, k = \overline{0, n-1}, v_n > t - S_{n-1} + u_{n+1}) =$$
$$= \sum_{n\ge1} \int_{-\infty}^{0} P(v_n > t - y) d_y\, P(S_k \le 0, k = \overline{0, n-1}, S_n - u_{n+1} \le y) =$$
$$= \sum_{n\ge1} \int_{-\infty}^{0} \bar{B}(t - y) d_y R_n(y) = \int_{-\infty}^{0} \bar{B}(t - y) dR(y),$$

where $v_n, n \ge 1$ the duration of the $n-$th customer's service time.

The last equality is true because of Theorem 6, Chapter 12, Section 2, [7].

V. THE MAIN RESULT

First of all, let us formulate auxiliary results.

Lemma 3. Let in the $GI|G|1|\infty$ model with FIFO discipline $\rho_1 < 1$ (see (2)) and DF B(x) of customer service times be IHR DF. Then the assumption: that W is an IHR DF implies that W^* also is an IHR DF.

Proof. It is easier to present with the help of the known Takac's formula

$$W^*(x) = 1 - \rho_1 + \rho_1 \cdot W(x) * \hat{B}(x), \quad (20)$$

where * denotes the sign of convolution, and

$$\hat{B}(x) = \frac{1}{\beta_1} \cdot \int_0^x \big(1 - B(u)\big) du. \text{ (compare to (13)).}$$

In [2], it was proved that if $B(x)$ is IHR DF, then $\hat{B}(x)$ is IHR DF too. According to Theorem 4.1. p.44, [1], the convolution of two IHR DF's without jumps at the point zero is an IHR DF. Examining the proof of this statement, we are convinced that the restriction on the absence of a jump at zero for IHR DF (see Definition 1) may be omitted.

Thus, in conditions of Lemma 3 $W(x)*\hat{B}(x)$ is an IHR DF. For any $x\epsilon R^+$ and $t\epsilon R^+$ from (20) we have

$$\frac{\bar{W}^*(x+t)}{\bar{W}^*(t)}=\frac{1-W(x+t)*\hat{B}(x+t)}{1-W(t)*\hat{B}(t)} \quad (21)$$

By Definition 1, since the convolution $W(x)*\bar{B}(x)$ *is IHR DF,* for any $x\epsilon R^+$, the right-hand-side of (21) doesn't increase with respect to t. Then the left-hand-side of (21) also doesn't increase with respect to t, i.e., W^* is an IHR DF.

Remark 3. Lemma 3 may be proved with the help of Hooke's formula too (see [4], Theorem 3.5, p. 158). Namely,

$W^*(x)=P(\max(0,w+v-\hat{u})\leq x),\ x\epsilon R^+,$

where *Rv's w, v, u* are independent and have DF's $W(x)$, $\bar{B}(x)$,

$$\hat{A}(x)=\frac{1}{\alpha_1}\int_0^x\big(1-A(u)\big)du.$$

Theorem 2. Let in the $GI\big|G\big|1\big|\infty$ model with FIFO discipline $\rho_1<1$ (see (2)) and DF B(x) of customer service times be IHR DF. Then DF's W and W* of two stationary waiting times are IHR DF's.

Proof. Due to *Lemma 3,* the result is enough to establish for W. Using *Lemma 2,* for $x\epsilon R^+$ and $t\epsilon R^+$ form the ratio

$$\frac{P(\xi_1>t+x)}{P(\xi_1>t)}=\frac{\int_{-\infty}^0\bar{B}(t+x+|y|)dR(y)}{\int_{-\infty}^0\bar{B}(t+|y|)dR(y)}. \quad (22)$$

We have to show that for the given $x\epsilon R^+$, the right-hand side in (22) doesn't decrease with respect to $t\epsilon R^+$. For any entire $n\geq 1$, let us devide the interval $(-n,0]$ on $m=n^n$ non-intersecting intervals of length $\frac{1}{n}$:

(-n, -n+$\frac{1}{n}$], (-n+$\frac{1}{n}$, -n+$\frac{2}{n}$], …(-$\frac{1}{n}$, 0].

The sequence $\{T_n(z)\}_1^\infty$ of functions with argument $z\epsilon R^+$ of the type

$$T_n(z)=\sum_{k=1}^m\bar{B}\left(z+\frac{k}{n}\right)\cdot c_{nk},\ n\geq 1, \quad (23)$$

where

$$0\leq c_{nk}=R\left(-\frac{k-1}{n}\right)-R\left(-\frac{k}{n}\right),k=\overline{1,m} \quad (24)$$

doesn't decrease with respect to $n\epsilon R^+$, and for each $z\epsilon R^+$, we have

$$\lim_{n\to\infty}T_n(z)=\int_{-\infty}^0\bar{B}(z+|y|)\,dR(y). \quad (25)$$

According to (22)-(25), for $t\epsilon R^+$ and $x\epsilon R^+$

$$\frac{P(\xi_1>t+x)}{P(\xi_1>t)}=\lim_{n\to\infty}\frac{T_n(t+x)}{T_n(t)}=\lim_{n\to\infty}\frac{\sum_{k=1}^m\bar{B}(t+x+\frac{k}{n})c_{nk}}{\sum_{k=1}^m\bar{B}(t+\frac{k}{n})c_{nk}}=$$

$$=\lim_{n\to\infty}\frac{\sum_{k=1}^m\bar{B}(t+x+\frac{k}{n})c_{nk}}{\sum_{k=1}^m\bar{B}(t+\frac{k}{n})c_{nk}}, \quad (26)$$

where

$$\alpha_k^x(t)=\frac{\bar{B}\left(t+x+\frac{k}{n}\right)}{\bar{B}\left(t+\frac{k}{n}\right)},\ \beta_k(t)=\bar{B}(t+\frac{k}{n}) \quad (27)$$

Let us show that for each $n\geq 1$, the expression at the right-hand side in (26) under the sign of limit doesn't increase with respect to *t.* Then, the left-hand side in (26), as a limit of non-increasing functions, will also be a non-increasing function with respect to *t.*

From the form of (27), and since $B(x)$ is IHR DF, it follows that $\alpha_k^x(t)$ and $\beta_k(t)$ for each $k=\overline{1,m}$ are non-increasing, and $\beta_k(t)\leq 1$.

First, assume that DF $B(x)$ has a density. Then $\alpha_k^x(t)$ and $\beta_k(t)$ for each $k=\overline{1,m}$ are differentiable with respect to *t.* Moreover,

$$\frac{d}{dt}(\alpha_k^x(t)\cdot\beta_k(t))\leq\frac{d}{dt}\beta_k(t))\leq 0,$$

$$\frac{d}{dt}(\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk})\leq$$

$$\leq\frac{d}{dt}(\sum_{k=1}^m\beta_k(t)\cdot c_{nk})\leq 0. \quad (28)$$

Further, $\frac{d}{dt}\{\frac{\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk}}{\sum_{k=1}^m\beta_k(t)\cdot c_{nk}}\}=$

$$=\{(\sum_{k=1}^m\beta_k(t)\cdot c_{nk})\cdot\frac{d}{dt}(\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk})-(\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk})\cdot\frac{d}{dt}(\sum_{k=1}^m\beta_k(t)\cdot c_{nk})\}/(\sum_{k=1}^m\beta_k(t)\cdot c_{nk}).^2$$

Estimate the numerator of the last expression denoted by $I(x,t)$. It's easy to see that

$$I(x,t)\leq\left(\sum_{k=1}^m\beta_k(t)\cdot c_{nk}\right)\cdot\frac{d}{dt}\left(\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk}\right)-$$

$$-\left(\sum_{k=1}^m\alpha_k^x(t)\cdot\beta_k(t)\cdot c_{nk}\right)\cdot\frac{d}{dt}\left(\sum_{k=1}^m\beta_k(t)\cdot c_{nk}\right)=0,$$

where (28) and inequalities $\beta_k(t)\leq 1,k=\overline{1,m}$ were used.

The statement in the case of $B(x)'s$ density existence is proved.

If DF $B(x)$ doesn't have density, then for each $n\geq 1$ consider the family of functions

$$T_n^\lambda(z)=\sum_{k=1}^m(1-B(z+\frac{k}{n})*$$

$$*(1-e^{-\lambda\cdot\left(z+f\frac{k}{n}\right)}))\cdot c_{nk},\lambda\epsilon R^+,z\epsilon R^+.$$

The DF $B_\lambda(t)=B(t)*(1-e^{-\lambda\cdot t})$, as a convolution of two IHR DF's, is an IHR DF. Moreover, there is a density of DF B_λ(t), and

$$\lim_{\lambda\to+\infty}T_n^\lambda(z)=T_n(z),\ z\epsilon R^+.$$

Thus,

$$\frac{T_n(t+x)}{T_n(t)}=\lim_{\lambda\to+\infty}\frac{T_n^\lambda(t+x)}{T_n^\lambda(t)}. \quad (29)$$

For entire n, $t\epsilon R^+, x\epsilon R^+$. Due to the above proof, we obtain that the expression at the right-hand side of (29) under the sign of limit doesn't increase with respect to *t.* Theorem 2 is proved.

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