Improvement of Growth Rate Bound of Stock Market with Side Information

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***Abstract*—The stock market is considered in case of the existence of some side information and a new bound for the increase in the growth rate is obtained using the concepts of logarithmically asymptotically optimal (LAO) testing of multiple hypotheses.**

***Keywords*— Portfolio, Optimal growth rate, Side information, Hypothesis testing, Reliabilities.**

I. Introduction

In the present paper, we consider a stock market model with multiple $K(\geq 1)$ stocks and the availability of some side information Y. The concepts of optimal portfolio and LAO testing of multiple hypotheses are used in the construction of a new bound for the increase of the growth rate.

The considered model and corresponding optimal portfolio problems were deeply investigated in the works of T. M. Cover and his coauthors (particularly in [1] – [8]). In these papers, the bound of the increase of the growth rate of continually distributed stock markets is represented.

The concept of LAO hypothesis testing is represented in the works of Haroutunian and his students (particularly in [9] – [11]), also in the important article of Ahlswede and Haroutunian in [12]. The application of the LAO hypothesis testing in portfolio theory is introduced in [13], where using error probabilities and reliabilities, a new, more exact bound for the increase of the growth rate of the stock market is found. The bound obtained in [13] is refined in [14] by using the LAO test existence theorem [9].

II. The Main Definitions and Notations

Let $\mathbf{X} = (X_1, X_2, ..., X_K)$, $X_k \geq 0$, $k = \overline{1,K}$, be a stock market with independent price relatives (the ratio of the price at the end of the day to the price at the beginning of the day) X_k, $k = \overline{1,K}$ defined on the finite set $\mathcal{X}$. So, typically, X_k is near 1. For example, $X_k = 1.03$ means that the i-th stock went up 3 percent that day.

We assume that probability distributions (PDs) of price relatives $X_k \geq 0\,(X_k \in \mathcal{X})$, $k = \overline{1,K}$, are unknown and R distinct possible PDs G_r, $r = \overline{1,R}$, are given for them. It is supposed that the stock prices are mutually independent. If the PD of the RV X_k is G_{m_k}, $m_k = \overline{1,R}$, $k = \overline{1,K}$, then for the stock market $\mathbf{X}$ the joint PD will be

$$G_m(\mathbf{x}) = \prod_{k=1}^{K} G_{m_k}(x_k),\ \mathbf{x} = (x_1, x_2, ..., x_K) \in \mathcal{X}^K,$$
$$m = (m_1, m_2, ..., m_K).$$

For a portfolio (an allocation of the wealth across the stocks) $\mathbf{b} = (b_1, b_2, ..., b_K)$, $b_k \geq 0$, $k = \overline{1,K}$, $\sum b_k = 1$, used for stock market $\mathbf{X}$, the wealth relative is $S = \mathbf{b}^t\mathbf{X}$, where $\mathbf{b}^t$ is the transposed vector of $\mathbf{b}$.

The maximum $W^*(G) = \max_{\mathbf{b}} W(\mathbf{b}, G)$ of the growth rate $W(\mathbf{b}, G) = \mathbf{E}(\log \mathbf{b}^t\mathbf{X})$ is called an optimal growth rate, where $\mathbf{b} = (b_1, b_2, ..., b_K)$ is the portfolio corresponding to the PD $G = G_m$. The argument $\mathbf{b}^* = \mathtt{Arg}(\max_{\mathbf{b}} W(\mathbf{b}, G))$, is called a log – optimal (or growth optimal) portfolio.

Let $\mathbf{x}_n = (x_{1,n}, x_{2,n}, \ldots, x_{K,n})$, $n = \overline{1,N}$, be a sample (results of independent observations), where $x_{k,n}$ is a result of the n – th realization of the price relative X_k, $k = \overline{1,K}$. The empirical PD of the stock market after N days will be:

$$G_m^{(N)}(\mathbf{x}^N) \ = \ \prod_{n=1}^{N} G_m(\mathbf{x}_n) = \prod_{n=1}^{N}\prod_{k=1}^{K} G_{m_k}(x_{k,n}).$$

In the result of the theorem it is used the mutual information between RVs $\mathbf{X}$ and $\mathbf{Y}$ with PDs $G(\mathbf{x})$ and $V(\mathbf{y})$

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correspondingly, and with joint PD $G(\mathbf{x},\mathbf{y})$:

$\mathbf{I}(\mathbf{X}\wedge\mathbf{Y}) = \sum\limits_{\mathbf{x}}\sum\limits_{\mathbf{y}} G(\mathbf{x},\mathbf{y})\log\frac{G(\mathbf{x},\mathbf{y})}{G(\mathbf{x})V(\mathbf{y})}$.

III. Problem Statement

LAO hypothesis testing will be used for the determination of unknown PD G_m on the base of the sample and will be applied in the construction of the bound for the increase of the growth rate.

The growth rate, when the incorrect portfolio $\mathbf{b}_l$ is used instead of the right portfolio $\mathbf{b}_m$ and some side information Y is available is the following

$$\Delta W = W(\mathbf{b}_m, G_m) - W(\mathbf{b}_l, G_m) \geq 0.$$

Each PD G_r from the set of the given PDs can be simultaneously accepted for some price relatives.

Let H_l be the hypothesis characterizing the possible joint PD: $H_l : G = G_l,\, l = (l_1, l_2, ..., l_K)$.

Let us define the non-randomized tests $\varphi^{(N)}$ by division of the sample space $\mathcal{X}^{KN}$ into R^K disjoint subsets

$$\mathcal{A}_l^{(N)} = \{\mathbf{x}^N : \varphi^{(N)}(\mathbf{x}^N) = l = (l_1, l_2, ..., l_K)\},$$

$$l_k = \overline{1,R}, k = \overline{1,K}.$$

Then the probability of the erroneous acceptance of hypothesis H_l provided that H_m is true, for $l \neq m$, is $\alpha_{l|m}^{(N)} = G_m^{(N)}(\mathcal{A}_l^{(N)})$. The probability of rejection of hypothesis H_m, when it is true is equal to $\alpha_{m|m}^{(N)} = \sum\limits_{l\neq m} \alpha_{l|m}^{(N)}$.

The matrix $\mathbf{E} = \{E_{l|m}\}$ of the sequence of tests φ, where

$$E_{l|m} = -\overline{\lim_{N\to\infty}}\frac{1}{N}\log\alpha_{l|m}^{(N)} \geq 0,\ \ l \neq m,$$

and $E_{m|m} = \min\limits_{l\neq m} E_{l|m}$, is called a matrix of reliabilities [9].

We call the test sequence φ^* LAO if for given positive values of a certain part of elements of the matrix of reliabilities $\mathbf{E}(\varphi^*)$ the procedure provides maximal values for all other elements of it.

Let $G^{(N)}\left(\mathbf{x}^N|\mathbf{Y}\right) \triangleq \prod\limits_{n=1}^{N} G(\mathbf{x}_n|\mathbf{Y})$, where $\mathbf{Y}$ is some side information with PD $V(\mathbf{y})$ and $G(\mathbf{x}|\mathbf{Y})$ is the corresponding conditional PD. We denote by $\alpha_m^{(N)}(\mathbf{Y}) = \sum\limits_{\mathbf{x}^N\in\mathcal{A}_m^{(N)}} G^{(N)}\left(\mathbf{x}^N|\mathbf{Y}\right)$ the error probability of using unconditional PD $G(\mathbf{x})$ instead of the conditional PD $G(\mathbf{x}|\mathbf{Y})$.

And the corresponding reliability will be defined as usual

$$E_m(\mathbf{Y}) \triangleq -\overline{\lim_{N\to\infty}}\frac{1}{N}\log\alpha_m^{(N)}(\mathbf{Y}).$$

Note, that we will have the portfolios of "maximal losses", if

$$\mathbf{b}_l^t\,\mathbf{x}\,G_l(\mathbf{x}) = \mathbf{b}_m^t\,\mathbf{x}\,G_m(\mathbf{x}),\ \mathbf{x}\in\mathcal{X}^K,\ l\neq m,\ \mathbf{b}_l \neq \mathbf{b}_m.$$

All other cases are the portfolios of "nonmaximal losses". The theorem of the next section represents the new bound for the increase in growth rate in case of the existence of side information and some statistical data (sample).

IV. The Result

Theorem: If the PDs $G(\mathbf{x}|\mathbf{Y})$ and $G(\mathbf{x})$ are different and the corresponding log – optimal portfolios $\mathbf{b}(\mathbf{Y}) \neq \mathbf{b}$ are portfolios of nonmaximal losses, then for each sample $\mathbf{x}^N = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)$ of the length N large enough, the increase in the growth rate ΔW due to side information $\mathbf{Y}$ is bounded by

$$\Delta W \leq I(\mathbf{X}\wedge\mathbf{Y}) + \frac{1}{N}\log\alpha_{m_1,m_2,...,m_K}^{(N)}(\mathbf{Y}). \qquad (1)$$

In case of maximal loses $\Delta W = I(\mathbf{X}\wedge\mathbf{Y})$.

Proof: Let $G(\mathbf{x},\mathbf{y})$ be the joint PD of $(\mathbf{X},\mathbf{Y})$. For the given side information $\mathbf{Y} = \mathbf{y}$, the investor must use the conditional log – optimal portfolio $\mathbf{b}(\mathbf{Y})$ for the conditional PD $G(\mathbf{x}|\mathbf{Y}=\mathbf{y})$. Hence, from Theorem proved in [13], for condition $\mathbf{Y}=\mathbf{y}$ and for large enough N, we have

$$\Delta W_{\mathbf{Y}=\mathbf{y}} \leq \sum_{\mathbf{x}\in\mathcal{X}^K} G(\mathbf{x}|\mathbf{Y}=\mathbf{y})\log\frac{G(\mathbf{x}|\mathbf{Y}=\mathbf{y})}{G(\mathbf{x})} + \frac{1}{N}\log\alpha_m^{(N)}(\mathbf{Y}=\mathbf{y}).$$

Summarizing this over possible values of $\mathbf{Y}$, we obtain

$$\Delta W \leq \sum_{\mathbf{y}} V(\mathbf{y})\sum_{\mathbf{x}\in\mathcal{X}^K} G(\mathbf{x}|\mathbf{Y}=\mathbf{y})\log\frac{G(\mathbf{x}|\mathbf{Y}=\mathbf{y})}{G(\mathbf{x})} + \frac{1}{N}\sum_{\mathbf{y}} V(\mathbf{y})\log\alpha_m^{(N)}(\mathbf{Y}=\mathbf{y}).$$

Using Jensen's inequality in [1] we get

$$\Delta W \leq \sum_{\mathbf{y}}\sum_{\mathbf{x}\in\mathcal{X}^K} V(\mathbf{y})G(\mathbf{x}|\mathbf{Y}=\mathbf{y})\log\left[\frac{G(\mathbf{x}|\mathbf{Y}=\mathbf{y})}{G(\mathbf{x})}\frac{V(\mathbf{y})}{V(\mathbf{y})}\right] + \frac{1}{N}\log\sum_{\mathbf{y}} V(\mathbf{y})\alpha_m^{(N)}(\mathbf{Y}=\mathbf{y}).$$

Due to total probability theorem

$$\sum_{\mathbf{y}} V(\mathbf{y})\alpha_m^{(N)}(\mathbf{Y}=\mathbf{y}) = \alpha_m^{(N)}(\mathbf{Y}).$$

So, we can deduce that

$$\Delta W \leq \sum_{\mathbf{y}}\sum_{\mathbf{x}\in\mathcal{X}^K} G(\mathbf{x},\mathbf{y})\log\frac{G(\mathbf{x},\mathbf{y})}{G(\mathbf{x})V(\mathbf{y})} + \frac{1}{N}\log\alpha_m^{(N)}(\mathbf{Y})$$

$$= I(\mathbf{X} \wedge \mathbf{Y}) + \frac{1}{N} \log \alpha_m^{(N)}(\mathbf{Y}).$$

The case of maximal losses is trivial.

Corollary 1: If the PDs $G(\mathbf{x}|\mathbf{Y})$ and $G(\mathbf{x})$ are different and the corresponding log – optimal portfolios $\mathbf{b}_m(\mathbf{Y}) \neq \mathbf{b}_m$ are portfolios of nonmaximal losses, then tending N to infinity in (1), *we obtain*

$$\Delta W \leq I(\mathbf{X} \wedge \mathbf{Y}) - E_m(\mathbf{Y}).$$

Corollary 2: The bound obtained in Theorem can be forced by using the LAO test existing theorem [9]- [11].

V. Conclusion

The bounds for the increase of growth rate of the stock market allow investors to assess the risk any time and change the current portfolio for limiting possible losses and obtain maximum incomes. In stock markets, the situation with side information is very important as various types of indices exist which can be used for the prediction of the trends of the stock market. That indices can also be used as side information in the construction of an optimal portfolio.

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